Reminder

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

1. converges!
2. $e^0 = I$
3. $\exp(\mathrm{diag}(\lambda_i)) = \mathrm{diag}(e^{\lambda_i})$
4. $AB = BA \Rightarrow e^{A+B} = e^A e^B$
5. $e^{(t+s)A} = e^{tA} e^{sA}$
6. $\frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA}A$

$\Rightarrow e^{At}y_0$ solves $\dot{y} = Ay$, $y(0) = y_0$

7. $e^{C^{-1}DC} = C^{-1}e^D C$

example

Solve $\begin{cases} \dot{x} = 4x - 6y & x(0) = 2 \\ \dot{y} = 3x - 5y & y(0) = -1 \end{cases}$ $\quad v = \begin{pmatrix} x \\ y \end{pmatrix}$

$\dot{v} = Av$, $v(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $A = \begin{pmatrix} 4 & -6 \\ 3 & -5 \end{pmatrix}$

$A \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow \exists$ basis v_1, v_2 of $\mathbb{R}^2$ s.t $\begin{cases} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \end{cases}$

singular $\rightarrow (A - \lambda_1 I)v_1 = 0$
$\rightarrow (A - \lambda_2 I)v_2 = 0$

$$\det(A - \lambda I) = \det\begin{pmatrix} 4-\lambda & -6 \\ 3 & -5-\lambda \end{pmatrix}$$
$$= \lambda^2 + \lambda - 2$$

$\lambda_{1,2} = -2, 1$

$(A - (-2)I)v_1 = 0 \Rightarrow \begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A - 1I)v_2 = 0 \Rightarrow \begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} v_2 = 0 \Rightarrow v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$A \overset{?}{=} CDC^{-1}$ ✓ $\quad C = (v_1 | v_2) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \; D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\overset{?}{=} \cancel{C^{-1}DC}$

$$
\begin{aligned}
v(t) &= e^{At}v_0 \\
&= e^{tCDC^{-1}}v_0 \\
&= Ce^{tD}C^{-1}v_0 \\
&= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \\
&\vdots
\end{aligned}
$$

This works whenever A has n distinct eigenvalues $\chi_A(\lambda) = \det(A - \lambda I)$ has n different roots. Otherwise,

<u>Theorem (The Jordan Canonical Form)</u>

In a vector space over $\mathbb{C}$ if $T : V \longrightarrow V$, then there is a basis $\beta = (v_1, \dots, v_n)$ of V, s.t.

$$D = [T]_\beta = \begin{pmatrix} \boxed{B_1} & & & 0 \\ & \boxed{B_2} & & \\ & & \boxed{B_3} & \\ 0 & & & \ddots \end{pmatrix} \quad B_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

$$\exp D = \begin{pmatrix} e^{B_1} & & 0 \\ & e^{B_2} & \\ 0 & & \ddots \end{pmatrix} \qquad \boxed{D = \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right) \Rightarrow D^2 = \left(\begin{array}{c|c} B_1^2 & 0 \\ \hline 0 & B_2^2 \end{array}\right)}$$

Enough to compute e^{tB_i}

$B_i = \lambda_i I + J$ where $J = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$

$$
\begin{aligned}
e^{tB_i} &= e^{t\lambda_i I + tJ} \\
&= e^{t\lambda_i I}e^{tJ} \\
&= e^{t\lambda_i} I e^{tJ}
\end{aligned}
$$

$J^2 = \begin{pmatrix} 0 & 0 & 1 & & 0 \\ & 0 & 0 & 1 & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ 0 & & & & 0 \end{pmatrix}, \; J^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ & \ddots & & \ddots & 1 \\ & & \ddots & & 0 \\ & & & \ddots & 0 \\ 0 & & & & 0 \end{pmatrix}$

$$(J^k)_{ij} = \begin{cases} 1 & j-i=k \\ 0 & \text{otherwise} \end{cases}$$

$$e^{tJ} = \sum \frac{t^k J^k}{k!}$$

$$= \begin{pmatrix} 1 & t & t^2/2 & t^3/6 & \cdots & \frac{t^{j-i}}{(j-i)!} \\ & 1 & t & t^2/2 & t^3/6 & \vdots \\ & & 1 & t & t^2/2 \;\; t^3/6 & \\ & & & \ddots & \ddots & t^3/6 \\ & \text{\Large 0} & & & \ddots & t^2/2 \\ & & & & \ddots & t \\ & & & & & 1 \end{pmatrix}$$

$$= \begin{cases} 0 & i > j \\ \frac{t^{j-i}}{(j-i)!} & i \le j \end{cases}$$

<u>example</u>

solve $\begin{cases} \dot{x} = 3x - y \\ \dot{y} = x + y \end{cases}$ $\begin{pmatrix} x(0) = 2 \\ y(0) = -1 \end{pmatrix} = v_0$ $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$

$A \sim \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$Av_1 = 2v_1$

$Av_2 = v_1 + 2v_2$

$(A-2I)v_1 = 0 \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A-2I)v_2 = v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$A = CDC^{-1}$

$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}$, $C = (v_1 | v_2)$