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$$y'' + py' + qy = 0 \overset{v_1 = y}{\underset{v_2 = y'}{\Longleftrightarrow}} v' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} v = Av, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$y_{1,2}$ indep. solutions. $W = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$ $\boxed{W' = (\mathrm{tr} A) W}$

Abel's Thm: $W' = -pW$

example

$y'' + y = 0$. $y_2 = \sin x$ $y_1 = \cos x$

$W' = -pW = 0 \therefore W = \text{Const}$

$$W(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\Rightarrow W(x) = 1 = \det \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = \cos^2 x + \sin^2 x$$

Problem

Find a power series.

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{s.t.} \quad y' = f(x, y) \quad y(x_0) = y_0$$

Motivations

1. Combinatorics
2. Power Series are lousy for large $(x - x_0)$ yet they are great approximations for small $(x - x_0)$

   Best example QED $\longrightarrow \frac{e^2}{\hbar \cdot c} \sim \frac{1}{137} \ll 1$

   "The fine structure constant"

Technique 1

$\phi_0 = y_0$

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t)) dt$$

$$|\phi_n(x) - \phi_{n+1}(x)| < |x - x_0|^{n+1} \cdot C$$

ϕ_n will give a power series approximation to degree n

Techique 2

$$\phi' = f(x, \phi) \quad \phi(x_0) = y_0$$

$$\phi = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

Substitute above ϕ into equation, differentiate $(k-1)$ times, evaluate at $x = x_0$

$$\text{LHS}: (\phi')^{(k-1)}\Big|_{x=x_0} = \phi^{(k)}\Big|_{x=x_0} = k!\,a_k$$

$$\text{RHS}: \frac{d^{k-1}}{dx^{k-1}} f\Big(x, \underline{\sum_{j=0}^{\infty} a_j (x-x_0)^j}\Big)\Big|_{x=x_0} = \frac{d^{k-1}}{dx^{k-1}} f\Big(x, \underbrace{\sum_{j=0}^{k-1} a_j (x-x_0)^j}_{\phi_{k-1}}\Big)$$

$$\phi_0 = y_0$$

$$\phi_k = \phi_{k-1} + a_k x^k$$

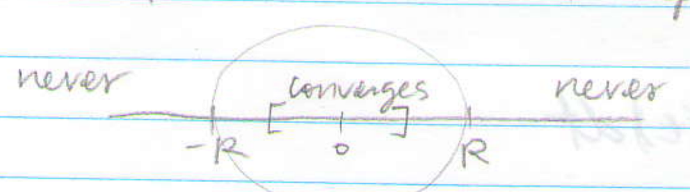
$$= \phi_{k-1} + \frac{x^k}{k!} \frac{d^{k-1}}{dx^{k-1}} f(x, \phi_{k-1})\Big|_{x=x_0}$$

Theorem 1

Given $\phi = \sum_{n=0}^{\infty} a_n x^n$ $\exists R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$

"radius of convergence of ϕ" s.t.

1. $\phi(x)$ absolutely converges if $|x| < R$
2. $\phi(x)$ diverges if $|x| > R$
3. If $|x| = R$ "it depends"



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proof

Let $R = \sup\{ r : a_n r^n \xrightarrow[n\to\infty]{} 0 \}$

$= \sup\{ r : |a_n r^n| \text{ is bounded} \}$

$|a_n r^n| < C$ take $r' < r$ and then

$$|a_n r'^n| = |a_n r^n| \left(\frac{r'}{r}\right)^n \le C \left(\frac{r'}{r}\right)^n \longrightarrow 0$$

proof of 1

Suppose $|x| < R \Rightarrow \exists r$ s.t. $|x| < r < R$ s.t. $a_n r^n \to 0$

but then,

$$|a_n||x|^n = |a_n| r^n \left(\frac{|x|}{r}\right)^n$$

$$< C \left(\frac{|x|}{r}\right)^n$$

So $\sum a_n x^n$ converges by comparison.