Mar 29

Thm: M^k compact orientable, $\omega \in \Omega^{k-1}(M) \Rightarrow \int_M d\omega = \int_{\partial M} \omega$

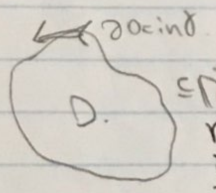
Example 0: $M = [0,1]_x \quad \omega \to f: [0,1] \to \mathbb{R} \quad \int_{[0,1]} f' = f(1) - f(0) \quad \partial[0,1] = (+,1) \cup (-,0)$

(*) "de-Rham complex" $\quad \Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3)$

$\{f\} \qquad \{\Sigma a_i dx_i\} \qquad \{b_1 dx_{23} + CP\} \qquad \{c\, dx_3\}$

$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$

vector calculus $\quad \{f\} \xrightarrow{\text{grad}} \left\{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right\} \xrightarrow{\text{curl}} \left\{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right\} \xrightarrow{\text{div}} \{c\}$.



$\gamma: [0,1] \to \mathbb{R}^2$
$\gamma(0) = \gamma(1)$.
$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$.

Example: $\omega = P\,dx + Q\,dy \quad P, Q: \mathbb{R}^2 \to \mathbb{R} \quad d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{\partial D} \underbrace{P\,dx + Q\,dy}_{} = \int_{[0,1]} (P \circ \gamma) \cdot \dot\gamma_1 + (Q \circ \gamma) \cdot \dot\gamma_2 = \int_{[0,1]} P \cdot \gamma_1 + Q \cdot \gamma_2$$

$\partial D = \operatorname{im}\gamma$

$\Omega^1[0,1] \ni \gamma^*(P\,dx + Q\,dy) = (P \circ \gamma) \cdot \dot\gamma_1\, dt + (Q \circ \gamma)\dot\gamma_2\, dt$. $\quad$ Pull back, so make sense on $[0,1]$.

green's thm.

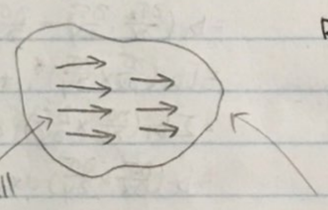
$F = \begin{pmatrix} Q \\ -P \end{pmatrix} \quad F_1 = Q. \quad F_2 = -P. \quad \text{LHS} = \int_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) = \int_D \operatorname{div} F$

rotation tangent to γ.

$\text{RHS} = \int (-F_2) \cdot \dot\gamma_1 + F_1 \dot\gamma_2 = \int \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \cdot \begin{pmatrix} \dot\gamma_2 \\ -\dot\gamma_1 \end{pmatrix} = \int \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot\gamma_1 \\ \dot\gamma_2 \end{pmatrix}$

$= \int_{[0,1]} F \cdot \vec{n} \, \|\dot\gamma\|$ where $\vec{n}$ is the unit normal to γD.

$= \int_{[0,1]} F \cdot \vec{n} \cdot \operatorname{vol}(\partial D) = \int_D \operatorname{div} F$ = total rain fall (divergence thm)



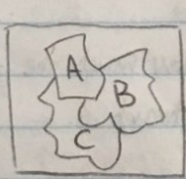
total outflow

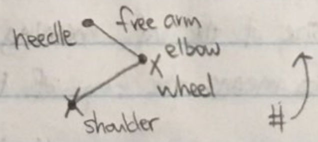
"how much fluid is created at every point"

Example 1: $\omega = \frac{1}{2}(x\,dy - y\,dx) \quad d\omega = dx \wedge dy \quad \int_D d\omega = \int_D 1 = \text{Area}(D) \overset{\text{Stock's}}{=} \int_{\partial D} \omega$.

$\Rightarrow$ you can compute the area of a domain by looking at its boundary.

A planimeter:





$\uparrow dy$
$\downarrow -dy$
$\rightarrow dx$
$\leftarrow -dx$

HW: Find the $\omega \in \Omega^1(\mathbb{R}^3)$ defined by a planimeter, compute $d\omega$.

Recall the "de-Rhamy complex" (*).	$\Omega^0(\mathbb{R}^3)$	$\Omega^1(\mathbb{R}^3)$	..	..	
vector calculus.	$\{f\}$	..	..	..	
M.	$\bigcup(S_i, P_i)$ sign. point.	$C = \operatorname{im}\gamma$ $\gamma: [0,1] \to \mathbb{R}^3$ "curve"	"surface" $S = \operatorname{im}\sigma$ $\sigma: D \to \mathbb{R}^3$	D. a closed domain.	

Integration: next time.

(Make up class) Mar 30.

Integration: $\int_M w$. $\Omega^0(\mathbb{R}^3)$ $\Omega^1(\mathbb{R}^3)$ $\Omega^2(\mathbb{R}^3)$

$\sum s_i F(p_i)$ 1 $\int a \cdot \dot r = \int a \cdot \vec T\, dv$ 2. $\int_S b \cdot \vec n\, dv$

Curve $0 \to 1$: $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \rightsquigarrow w = \sum a_i dx_i$, $r_i = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$ evaluated at r_i

$\int_T w = \int_{[0,1]} r^* w = \int_{[0,1]} \sum a_i dr_i = \int_{[0,1]t} \sum a_i \dot r_i\, dt = \int_{[0,1]} a \cdot \dot r \overset{*}{=} \int_{[0,1]} a \cdot \vec T \cdot \|\dot r\| = \int_{[0,1]} a \cdot \vec T\, dv$.

vector

$*$: if we write $\dot r = \|\dot r\| \cdot \vec T$ ← the unit tangent to T

k=1

$\boxed{1}$ $\int_{T} (\text{grad } F) \cdot \vec T\, dv = F(r(1)) - F(r(0))$

curve T

function F

increase direction to whole curve

↖ Integral of "Total climb rate"

Surface $1 \to 2$: $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \rightsquigarrow w = b_1 dx_2$, 3+cp, $\sigma = \begin{pmatrix} \sigma_1(x,y) \\ \sigma_2(x,y) \\ \sigma_3(x,y) \end{pmatrix}$

$\int_S w = \int_{D^2_{x,y}} \sigma^* w = \int_{D^2} b \cdot \vec n\, v\left(\frac{\partial\sigma}{\partial x}, \frac{\partial\sigma}{\partial y}\right) = \int_S b \cdot \vec n\, dv$. $\boxed{\#}$

$\sigma^* w = \sigma^*(b_1 dx_2 \wedge dx_3 + c.p.) = b_1 d\sigma_2 \wedge d\sigma_3 + cp = b_1\left(\frac{\partial\sigma_2}{\partial x} dx + \frac{\partial\sigma_2}{\partial y} dy\right) \wedge \left(\frac{\partial\sigma_3}{\partial x} dx + \frac{\partial\sigma_3}{\partial y} dy\right) + c.p.$

$= b_1\left(\frac{\partial\sigma_2}{\partial x} \cdot \frac{\partial\sigma_3}{\partial y} - \frac{\partial\sigma_3}{\partial x} \cdot \frac{\partial\sigma_2}{\partial y}\right) dx \wedge dy + c.p.$

$= [b_1 (\frac{\partial\sigma}{\partial x} \times \frac{\partial\sigma}{\partial y})^*_1 + c.p.]\, dx \wedge dy$

$= \sum b_i (\frac{\partial\sigma}{\partial x} \times \frac{\partial\sigma}{\partial y})_i\, dx \wedge dy$

$= b \cdot (\frac{\partial\sigma}{\partial x} \times \frac{\partial\sigma}{\partial y})\, dx \wedge dy$

$= b \cdot \vec n \cdot v(\frac{\partial\sigma}{\partial x}, \frac{\partial\sigma}{\partial y})\, dx \wedge dy$

$*: \begin{pmatrix} \frac{\partial\sigma_1}{\partial x} \\ \frac{\partial\sigma_2}{\partial x} \\ \frac{\partial\sigma_3}{\partial x} \end{pmatrix} \times \begin{pmatrix} \frac{\partial\sigma_1}{\partial y} \\ \frac{\partial\sigma_2}{\partial y} \\ \frac{\partial\sigma_3}{\partial y} \end{pmatrix}$

(where $v_1 \times v_2 = v(v_1, v_2) \cdot \vec n$ the unit normal to v_1 & v_2). ↖ unit normal to S.

$\frac{\partial\sigma}{\partial x}, \frac{\partial\sigma}{\partial y}$ pushes forwards of the std e_1, e_2 using σ.

Visualise $\boxed{\#}$ flow in $\mathbb{R}^3$, surface in $\mathbb{R}^3$. surface has normal at any point, and take normal, take component at the direction at normal.

V.f.L.S $\Rightarrow 0$. V.f.L.S $\Rightarrow$ the size of the v.f. meaning how much the flow "flowes" through the surface.

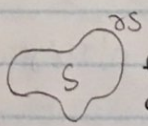
So it means the flow of b through s.

k=2.

$\boxed{2}$ $\int_S (\text{curl } a) \cdot \vec n\, dv = \int_S dw = \int_{\partial S} w = \int_{\partial S} a \cdot \vec T\, dv$

surfaces

v.f. a. ← since we use a in $\Omega^1(\mathbb{R}^3)$



flow in $\mathbb{R}^3$, = a

curl a measures # flow tends to spin.

Each spin measures only the part of spining, whose access is ⊥ to the surface.

LHS: At around 5.55, WATCH THE VIDEO. Local spining in the plane of s.

RHS: Easier. flows parallel to the boundary. Circulation of a along ∂S.

hold in any dim → thm split in different dim.

$\Omega^0 \xrightarrow{} \Omega^1 \xrightarrow{} \Omega^2 \xrightarrow{?} \Omega^3$ ← Gauss

Fund thm of calculus — original: stock's surface s vector field a.

total elevation gain $= \int_{hin}$ climbrato. $=$ circulation of a around ∂S.

$= \int_S$ normal component of curl a.

Inputs b: V.F. c: function. D: a domain/solid in $\mathbb{R}^3$ S: surface.

$\omega_2(b) = b_1 dx_2 \wedge dx_3 + \text{c.p.}$ $\omega_3(b) = c\, dx_1 \wedge dx_2 \wedge dx_3$. $d\omega_2(b) = \omega_3(\operatorname{div} b)$.

$\int_S \omega_2(b) = \int_S b\vec{n}\, dv$ $\int_D \omega_3(c) = \int_D c$.

$\int \operatorname{div} b = \int_D \omega_3(\operatorname{div} b) = \int_D d\omega_2(b) = \int_D d\omega = \int_{\partial D} \omega = \int_{\partial D} \omega_2(b) = \int_{\partial D} b\cdot\vec{n}\, dv$.

$\omega = \omega_2(b)$.

Gauss's thm / divergence thm.

Let $\omega \in \Omega^k(M)$ 1. ω is closed if $d\omega = 0$. → the integral of ω on a boundry is 0: If $N^{k+1} \subset M$ $\int_{\partial N} \omega = \int_N d\omega = 0$.

2. ω is "exact" if $\exists \lambda \in \Omega^{k-1}(M)$ s.t. $\omega = d\lambda$.

(If $N^k \subset M$ & $\partial N = \emptyset$ $\int_N \omega = \int_N d\lambda \overset{\text{stoke's}}{=} \int_{\partial N} \lambda = 0$.

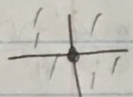
$\Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1}$ ω is closed $\Leftrightarrow \omega \in \ker d|_{\Omega^k}$ ω is exact $\Leftrightarrow \omega \in \operatorname{im} d|_{\Omega^{k-1}}$

Comment: Every exact form is closed.

pf: ω exact $\Rightarrow \omega = d\lambda$ for some λ, $d\omega = d(d\lambda) = 0$ □

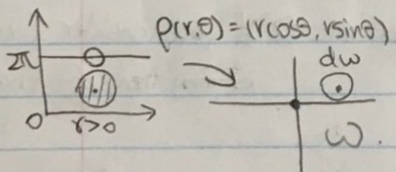
Example: $\omega = \frac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$

ω is closed: 1. $d\omega = \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right) dx \wedge dy = 0$.



2. $p^*(d\omega) = d p^* \omega = d\, p^* \frac{r^2(\cos^2\theta + \sin^2\theta)}{r^2} = d(d\theta) = 0$.

$\Rightarrow d\omega$ is 0 because near every $p \in \mathbb{R}^2_{r,\theta}$, p is invertible.



3. $\omega = d(\arctan\frac{y}{x}) = \arctan(\frac{-x}{y}) \pm \frac{\pi}{2}$

$\Rightarrow d\omega = d(d-) = 0$ except if $x = 0$.

ω is not exact (but nearly so) $\omega = d\,(\overbrace{\arctan\frac{y}{x}}^{\lambda})$ nearly, but λ not fully defined.

Indeed, $\int_{S^1} \omega = \int_{S^1} x\,dy - y\,dx = 2\pi \neq 0$.

Poincare's lemma: on $\mathbb{R}^n$ every closed form is exact.

de-Rham If M is compact, closed $\xrightarrow[\text{exact}]{\text{nearly}}$ $H^k_{dR}(M) = \underbrace{\left\{\begin{matrix}\text{closed}\\ k\text{-forms}\end{matrix}\right\} / \left\{\begin{matrix}\text{exact}\\ k\text{-forms}\end{matrix}\right\}}_{\text{Always finite dim.}}$