Mat 257 Oct 14, 2016 (1/1)

$$f(a+h) = f(a) + Df(a)\cdot h + o(h)$$

Class C^r: The first r derivatives exist and are cts.

<u>Thm</u>: If $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is C^2, then $\partial_i := \frac{\partial}{\partial x_i}$

$$\partial_i \partial_j f = \partial_j \partial_i f$$

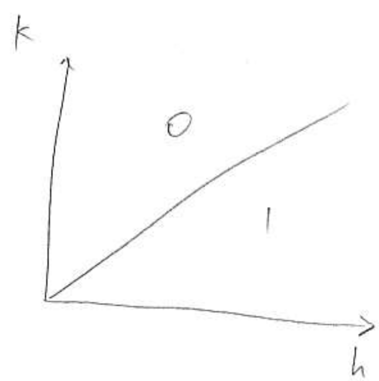
<u>e.g</u>:

$$\begin{tikzcd}
 & x^2 y \arrow[dl, "\partial x"'] \arrow[dr, "\partial y"] & \\
2xy \arrow[dr, "\partial y"'] & & x^2 \arrow[dl, "\partial x"] \\
 & 2x &
\end{tikzcd}$$

<u>e.g</u>: $f(h,k)$ s.t.

$$\lim_{h\to 0}\lim_{k\to 0} f(h,k) \neq \lim_{k\to 0}\lim_{h\to 0} f(h,k)$$

$$f(h,k) = \begin{cases} 0 & h \le k \\ 1 & h > k \end{cases}$$



$$1 = \underbrace{\lim_{h\to 0}\lim_{k\to 0} f(h,k)}_{=1}$$

$$\neq \underbrace{\lim_{k\to 0}\lim_{h\to 0} f(h,k)}_{=0}$$

$$= 0$$

Lem: $\forall h, \exists p, q \in [x_0, x_0+h] \times [y_0, y_0+h]$

$$\begin{tikzcd}
\partial_x \partial_y f(p) = \arrow[dr, "p \to \binom{x_0}{y_0}"] & \dfrac{f(x_0,y_0) - f(x_0+h,y_0) - f(x_0,y_0+h) + f(x_0+h,y_0+h)}{h^2} \arrow[d, "h \to 0"] & = (\partial_y \partial_x f)(q) \arrow[dl, "q \to \binom{x_0}{y_0}"'] \\
 & (\partial_x \partial_y f)\binom{x_0}{y_0} \overset{!}{=\!=} (\partial_y \partial_x f)\binom{x_0}{y_0} &
\end{tikzcd}$$

if φ cts,

$$\lim_{h\to 0} \lim_{k \to 0} \varphi(h,k) = \lim_{k\to 0} \lim_{h\to 0} \varphi(h,k)$$

$$\varphi(h,k) = \frac{f(x_0,y_0) - f(x_0+h,y_0) - f(x_0,y_0+h) + \cdots}{h \cdot k}$$

pf: Let $g(x) = \dfrac{f(x, y_0+h) - f(x, y_0)}{h}$, $\lambda = \dfrac{g(x_0+h) - g(x_0)}{h}$

By MVT, $\exists x_1 \in [x_0, x_0+h]$ s.t.

$$\lambda = g'(x_1)$$

$$= \frac{\partial_x f(x_1, y_0+h) - \overbrace{\partial_x f(x_1, y_0)}^{z}}{h}$$

$$z(y) = \partial_x f(x_1, y)$$

By MVT for z, $\exists y_1$ $\rightsquigarrow$ $= \partial_y z(y_1)$

$$= \partial_y \partial_x f(x_1, y_1)$$

Set $p = \binom{x_1}{y_1}$. Do the same with $x \longleftrightarrow y$.

Find q

$\square$