Nov 20, 2012

$$\sum_{k=0}^{n} y^{(k)}(x) P_k(x) = 0, \quad P_n(x) = 1$$

If P_k are analytic in nbd of 0. <u>ordinary point</u>
If not ordinary but $x P_k$ are analytic in nbd of 0, then <u>regular singular point</u>

<u>example</u>

$x^2 y'' + xy' = y$ regular singular
$x^3 y' = (x+1) y$ regular singular

If $x=0$ is regular singular then $\exists$ solution of form $x^\alpha A(x)$

<u>example</u>

$$y'' + \frac{y}{4x^2} = 0$$

If $y = \sum_{n=0}^{\infty} a_n x^n$ then we get $x^n : (n-\frac{1}{2})^2 a_n = 0, \ n = 0, 1, 2, \cdots$

$\Rightarrow a_n = 0 \ \forall n$. so $y(x) = 0$

Instead try $y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n$

$x^{n+\alpha} : [(n+\alpha)(n+\alpha-1) + \frac{1}{4}] a_n = 0, \ n = 0, 1, 2, \cdots$

$(\alpha(\alpha-1) + \frac{1}{4}) a_0 = 0, \ \alpha = \frac{1}{2}$

$\Rightarrow a_1 = 0, \ a_2 = 0, \ \cdots$

$y(x) = x^\alpha a_0 = a_0 \sqrt{x}$

<u>example</u>

$y'' + \frac{P(x)}{x} y' + \frac{q(x)}{x^2} y = 0$ and it has a regular singular point at $x=0$

So p, q are analytic at $x=0$

$$p(x)=\sum_{n=0}^{\infty}p_n x^n,\quad q(x)=\sum_{n=0}^{\infty}q_n x^n$$

Try $y(x)=\sum_{n=0}^{\infty}a_n x^{n+\alpha}$

$$y'=\sum_{n=0}^{\infty}(n+\alpha)a_n x^{n+\alpha-1},\quad y''=\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2}$$

$$\Rightarrow \sum_{n=0}^{\infty}a_n(\alpha+n)(\alpha+n-1)x^{\alpha+n}+\sum_{n=0}^{\infty}p_n x^n\sum_{n=0}^{\infty}a_n(\alpha+n)x^{\alpha+n}$$

$$+\sum_{n=0}^{\infty}a_n x^n\sum_{n=0}^{\infty}a_n x^{n+\alpha}=0$$

x^{α} : $a_0\alpha(\alpha-1)+p_0 a_0 x+q_0 a_0=0$

$P(\alpha)=\alpha^2+(p_0-1)\alpha+q_0$

indicial polynomial

$x^{n+\alpha}$: $a_n(\alpha+n)(\alpha+n-1)+\sum_{k=0}^{n}p_{n-k}\,a_k(\alpha+k)+\sum_{k=0}^{n}q_{n-k}a_k=0$

$$\Rightarrow P(\alpha+n)a_n=-\sum_{k=0}^{n-1}a_k[(\alpha+k)p_{n-k}+q_{n-k}]$$

If $P(\alpha+n)\neq 0$ for any $n=1,2,3,\dots$ then a_n is determined.

Let α_1, α_2 be roots of $P(\alpha)$. $\operatorname{Re}\alpha_1\geq \operatorname{Re}\alpha_2$.

Then $P(\alpha_1+n)\neq 0$ for $n=1,2,3,\dots$

If $y(x)=x^{\alpha_1}\sum_{n=0}^{\infty}a_n x^n$ then a_n can be determined so that

y is a solution.

To find a second solution there are four possibilities

1) If $\alpha_1 \neq \alpha_2 + n$ for any $n = 1, 2, \ldots$
then we can determine the coefficients of
$$y_2(x) = x^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n$$
$$y_1 = x^{\alpha_1} a_0 + \cdots$$
$$y_2 = x^{\alpha_2} b_0 + \cdots$$

2) If $\alpha_1 = \alpha_2 + n$ for some $n = 1, 2, 3, \ldots$
but $\sum_{k=0}^{n-1} a_k [(\alpha+k) p_{n-k} + q_{n-k}] = 0$
then a_n is free
$$P(\alpha+n) a_n = \Sigma$$
There is an independent solution of form
$$x^{\alpha_1 + N} \sum_{n=0}^{\infty} b_n x^n$$

- $\alpha_1 = \alpha_2$
- $\alpha_1 - \alpha_2 = n > 0$

$$\sum_{k=0}^{n-1} a_k [(\alpha+k) p_{n-k} + q_{n-k}] \neq 0$$

In these cases. second solution does not have form
$$x^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n$$

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3) If $\alpha_1 = \alpha_2$
If $y(x) = x^{\alpha_1} \sum_{n=0}^{\infty} a_n x^n$ solves $y'' + \frac{p}{x} y' + \frac{q}{x^2} y = 0$
then $y(x, \alpha) = x^{\alpha} \sum_{n=0}^{\infty} a_n(\alpha) x^n$, $a_n(\alpha_1) = a_n$

Let $L = \frac{d^2}{dx^2} + \frac{P(x)}{x} \frac{d}{dx} + \frac{q}{x^2}$

$(Ly_1) = 0$

$(Ly)(x, \alpha) = a_0 x^{\alpha-2} P(\alpha)$

$\frac{d}{d\alpha}\Big|_{\alpha=\alpha_1}$, $RHS = a_0 x^{\alpha-2} \log x P(\alpha) + a_0 x^{\alpha-2} P'(\alpha) = 0$

$$LHS = \frac{d}{d\alpha}\left[(Ly)(x, \alpha)\right]\Big|_{\alpha=\alpha_1}$$

$$= L\left[\left(\frac{d}{d\alpha} y\right)\right](x, \alpha_2)$$

$$\frac{d}{dx} y(x, \alpha) = \sum_{n=0}^{\infty} a_n'(\alpha) x^{n+\alpha} + a_n(\alpha) x^n x^\alpha \log x$$

Let $b_n = \left[\frac{d}{d\alpha} a_n(\alpha)\right](\alpha_2)$

Then $\left(\frac{dy}{d\alpha}\right)(x, \alpha_2) = y(x, \alpha_1) \log x + \sum_{n=0}^{\infty} b_n x^{n+\alpha_1}$

$$y_2(x) = \log x\, y_1(x) + \sum_{n=0}^{\infty} b_n x^{n+\alpha_1}$$

4) $\alpha_1 = \alpha_2 + n$ for $n > 0$