$y'' + py' + qy = 0$. $z = v(x)$ with $v'' + pv' = 0$

$\Downarrow$

$$\frac{d^2y}{dz^2} + Q(z)y = 0 \quad \text{with } Q = \frac{q}{(v')^2}$$

example

$y'' - \frac{2}{x}y' + y = 0$, $y'' + x^\alpha y = 0$

$v'' - \frac{2}{x}v' = 0 \qquad \mu = v'$

$x\mu' - 2\mu = 0$

$v' = \mu = x^2$

$v = \frac{x^3}{3} = z$

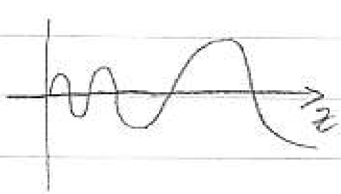
$x = \sqrt[3]{3z}$

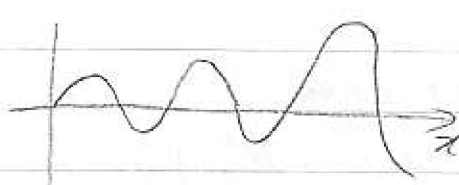
$$\frac{d^2y}{dz^2} + Q(z)y = 0.$$

$$Q = \frac{q}{(v')^2} = \frac{1}{(x^2)^2} = \frac{1}{(\sqrt[3]{3z})^4} = \frac{1}{(3z)^{4/3}} > \frac{1}{z^2} \text{ for large } z$$

Oscillate?

By comparison with $\frac{1}{z^2}$, oscillates





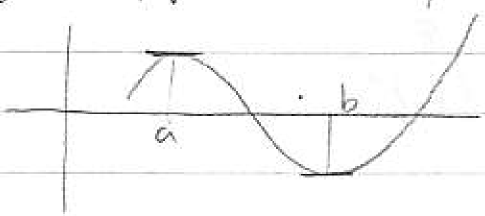
Handout page 8

Theorem 6.1

$y'' + py' + qy = 0$. $q > 0$

$y'(a) = 0 = y(b)$ $a < b$

$q' + 2pq > 0 \Rightarrow |y(a)| > |y(b)|$ "amplitudes decrease"
$q' + 2pq < 0 \Rightarrow |y(a)| < |y(b)|$ " " increase"



proof

Consider $F = y^2 + \frac{(y')^2}{q}$

$$F' = 2yy' + \frac{q2y'y'' - (y')^2 q'}{q^2}$$

$$= 2yy' + \frac{2qy'(-py' - qy) - q'(y')^2}{q^2}$$

$$= \frac{1}{q^2}\left(-q'(y')^2 - 2pq(y')^2\right)$$

$$= -\frac{(y')^2}{q^2}(q' + 2pq)$$

If $q' + 2pq > 0$. then $F' < 0$. So F is decreasing

$F(a) = (y(a))^2 + 0 > F(b) = (y(b))^2$

$|y(a)| > |y(b)|$

example

Bessel's equation

$$y'' + \underbrace{\frac{1}{x}}_{p} y' + \underbrace{\left(1 - \frac{\alpha^2}{x^2}\right)}_{q} y = 0$$

$$q' + 2pq = \frac{2\alpha^2}{x^3} + \frac{2}{x}\left(1 - \frac{\alpha^2}{x^2}\right) = \frac{2}{x} > 0$$

"amplitudes decrease"

Dec 4, 2012

Theorem 6.2

$y'' + py' + qy = 0$, $\quad q > 0 \quad y'(a) = 0 = y'(b) \quad a<b$

Let P be an an anti-derivative of p: $P' = p$

Then,

$$q' + 2pq > 0 \Rightarrow e^{P(a)}\sqrt{q(a)}\,|y(a)| < e^{P(b)}\sqrt{q(b)}\,|y(b)|$$

$$\text{''} \quad < 0 \Rightarrow \qquad \text{''} \qquad > \qquad \text{''}$$

$$q' + 2pq > 0 \Rightarrow 1 < \left|\frac{y(a)}{y(b)}\right| < e^{P(b)-P(a)}\sqrt{\frac{q(b)}{q(a)}}$$

proof

Consider $G = e^{2P}(qy^2 + (y')^2)$

$q' + 2pq > 0 \Rightarrow G' > 0$. G is increasing

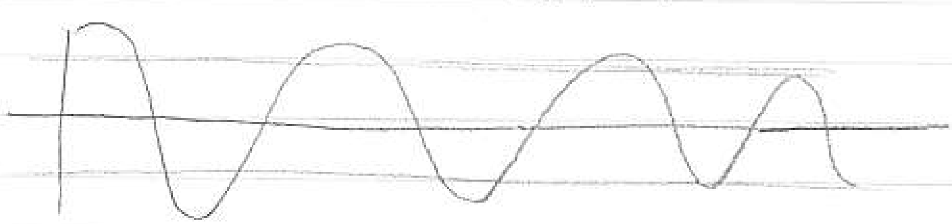
$G(a) = (LHS)^2 < G(b) = (RHS)^2$

Corollary

Consider $y'' + qy = 0$

If $q \xrightarrow[x \to \infty]{\text{monotone}} L > 0$ Then.

the amplitudes of the oscillations of y converge to a limit in $(0, \infty)$



proof

Assume $q' > 0$ (the other case is the same)

$$1 < \underbrace{\left|\frac{y(a)}{y(b)}\right|}_{\xrightarrow{a,b\to\infty} 1} < \underset{\parallel}{\cancel{e^{P(b)-P(a)}}}\sqrt{\frac{q(b)}{q(a)}} \xrightarrow[b,a\to\infty]{} 1$$

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\alpha^2}{x^2}\right)y = 0 \qquad v = \sqrt{x}\,y$$

$$\Rightarrow v'' + \left(1 - \frac{1 - 4\alpha^2}{4x^2}\right)v = 0$$

$$\Big\downarrow^{\text{monotone}}_{\ x \to \infty}$$

$$1$$

oscillation of v look like constant amplitude

oscillation of $y = \frac{v}{\sqrt{x}}$ approach 0 like $\frac{1}{\sqrt{x}}$