e.g: $f(x) = c$ is integrable on Q

pf: Take $\underline{P} = (a_1 = t_{10} < t_{11} = b_1 \,;\, a_2 < b_2 \,;\, \cdots \,;\, a_n < b_n)$

$\underline{P} = \{Q\}$

$L(f, \underline{P}) = c \cdot v(Q)$

$U(f, \underline{P}) = c \cdot v(Q)$

$U - L = 0 < \varepsilon \Rightarrow f$ integrable.

e.g: $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

$m_R(f) = 0$ same as previous example with $c = 0$

$M_R(f) = 1$ " " $c = 1$

$a < b$. $Q = [a, b]$

$L(f, P) = 0$

$U(f, P) = 1 \cdot (b - a) = b - a$

$U(f, P) - L(f, P) = b - a$
$\not< \varepsilon$

f is not integrable.

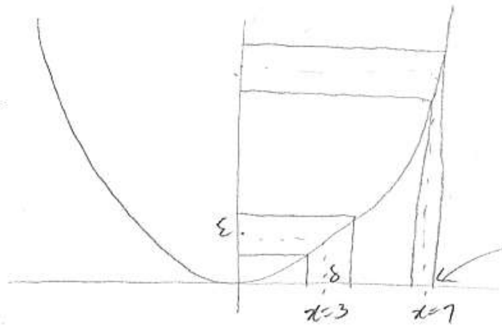
Def: f is cts on Q means

"$\forall x \in Q, \forall \varepsilon > 0, \exists \delta > 0. \forall y \in Q, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$"

f is unif. cts on Q means

"$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x, y \in Q, \; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$"

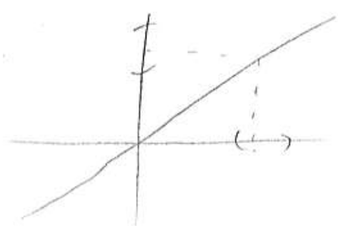
e.g: $f(x) = x^2$ on R



δ for 7 is much less than δ for 3

f is cts but not unif. cts.

e.g: $f(x) = x$



unif. cts.

Thm1: Every cts fct on Q is integrable

Thm2: Every unif. cts fct is integrable.

Thm3: Every cts fct on a cpt set is unif. cts.

pf of Thm 2

Sps f unif. cts on $Q \subset \mathbb{R}^n$

Let $\varepsilon > 0$ be given. By unif. cty. find $\delta > 0$ s.t.

if $x, y \in Q$. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2\mathrm{vol}(Q)}$.

Find a very fine partition P of Q s.t.

if $R \in P$ & $x, y \in R$. $|x-y| < \delta$.

Now for any $R \in \underline{P}$, $M_R(f) - m_R(f) = \sup_R f(x) - \inf_R f(x)$

$$\leq \frac{\varepsilon}{vol(Q)}$$

$$\begin{aligned} U(f,\underline{P}) - L(f,\underline{P}) &= \sum_R v(R)\,(M_R(f) - m_R(f)) \\ &\leq \sum_R v(R) \frac{\varepsilon}{2\,vol(Q)} \\ &= \frac{\varepsilon}{2\,vol(Q)} \sum_R v(R) \\ &= \frac{\varepsilon}{2\,vol(Q)} \cdot vol(Q) \\ &= \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

So by Riemann, f is integrable on Q $\square$