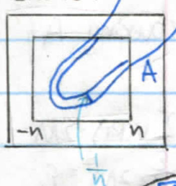
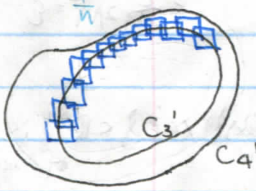
Sketch:



Claim: For any A, can find C_n s.t. $C_n \nearrow A \subset \mathbb{R}^k$

So set $C_n' = \underbrace{[-n,n]^k}_{\text{closed \& bdd}} \cap \underbrace{\{x : d(x,A^c) \geq \frac{1}{n}\}}_{\text{closed}}$ (compact) (ϕ cont) $\quad \{x : \phi(x) \geq \frac{1}{n}\} = \phi^{-1}([\frac{1}{n},\infty))$ closed, closed.



easy $C_n' \subset \text{int } C_{n+1}'$, $\bigcup_n C_n' = A$. Then get the C_n's as follows:

← Replace each C_n' by a finite union of rectangles that cover it and are contained in $\text{int } C_{n+1}'$ (using compactness).

proof (thm before) Assume $f \geq 0$, if $\fint_A f$ exists then $\forall n$, $\fint_{C_n} f \leq \fint_A f$ is an inc. seq. so $\fint_{C_n} f = \int_{C_n} |f|$ has a limit.

Nov. 28

$f \geq 0 \Rightarrow \fint_A f := \sup\{\fint_D f : D \subset A$ compact & rectifiable$\}$ (when sensible)

otherwise, $\fint_A f := \fint_A f_+ - \fint_A f_-$ (when sensible) $\quad y = \frac{1}{x}$

Def. $C_n \nearrow A$ "C_n rises to A" 1. C_n is compt & rect 2. $C_n \subset \text{int } C_{n+1}$ 3. $\bigcup C_n = A$.

Thm. Given $C_n \nearrow A$, $\fint_A f$ exists iff $(\fint_{C_n} |f|)$ is bdd and then $\fint_A f = \lim_{n\to\infty} \int_{C_n} f$

pf. Assume $f \geq 0$, $\fint_A f$ exists $\Rightarrow \fint_{C_n} |f| \leq \fint_A f$ so $\lim_{n\to\infty} \int_{C_n} f$ exists and is $\leq \fint_A f$

If $\lim_{n\to\infty} \int_{C_n} f$ exists and $D \subset A$, is compact & rectifiable then $D \subset \bigcup C_n \subseteq \bigcup \text{int } C_n$

by compactness $\exists N$ s.t. $D \subset \bigcup_{n=1}^{N} \text{int } C_n = \text{int } C_N \subset C_N$.

$\Rightarrow \fint_D f \leq \int_{C_N} f \leq \lim_{N\to\infty} \int_{C_N} f \Rightarrow \fint_A$ exists $\Rightarrow \fint_A f \leq \lim_{n\to\infty} \int_{C_n} f$

$\Rightarrow \fint_A f = \lim_{n\to\infty} \fint_{C_n} f$

Otherwise $f = f_+ - f_-$, $f_+, f_- \geq 0$, $f_+ + f_- = |f|$, ~~then~~

then $\fint_A f$ exists $\Leftrightarrow \fint_A f_+$ & $\fint_A f_-$ exists $\Leftrightarrow \lim_{n\to\infty} \fint_{C_n} f_+$ & $\lim_{n\to\infty} \fint_{C_n} f_-$ exists

$\Rightarrow \lim_{n\to\infty} \fint_{C_n} f_+ + \fint_{C_n} f_- = \lim_{n\to\infty} \fint_{C_n} f_+ + f_- = \lim_{n\to\infty} \fint_{C_n} |f|$ exists.

(Also if $\int_{C_n} |f|$ is bdd, then so are $\int_{C_n} f_+ \cdot \int_{C_n} f_-$ as $f_+ \leq |f|$ & $f_- \leq |f|$.)

So $\fint_A f$ exists $\Leftrightarrow \lim_{n\to\infty} \fint_{C_n} |f|$ exists. In that case, $\lim_{n\to\infty} \fint_{C_n} f = \lim_{n\to\infty} \fint_{C_n} f_+ - f_- = \lim_{n\to\infty} \fint_{C_n} f_+ - \lim_{n\to\infty} \fint_{C_n} f_-$

$= \fint_A f_+ - \fint_A f_- = \fint_A f$ □.

Thm. $\fint$ have good properties
i) $\fint_A af + bg = a \fint_A f + b \fint_A g$.
ii) $f \leq g$ $\fint_A f \leq \fint_A g$. & $|\fint_A f| \leq \fint_A |f|$.
iii) If $B \subset A$ & $f \geq 0$, $\fint_B f \leq \fint_A f$.
iv). If A & B are both open & f is integrable on both, then $\fint_{A\cup B} = \fint_A f + \fint_B f - \int_{A\cap B} f$.

Thm. If $A \subset \mathbb{R}^n$ is bdd & open & $f: A \to \mathbb{R}$ is bdd & cont. then $\fint_A f$ exists, and if $\int_A f$ also exists, then $\fint_A f = \int_A f$.

proof. If $D \subset A$ is compact & rectifiable. $\oint_D |f| \le \int_Q M = M \cdot \text{vol}(Q)$

$D \subset Q$. Q is a rectangle. so $\int_{C_n} |f|$ is bdd. & $|f| \le M$ integrable ~~everywhere~~ whenever $C_n \uparrow A$

So $\fint_A f$ exists. □

Nov. 30.

$A \subseteq \mathbb{R}^k$ open, $f: A \to \mathbb{R}$ cont. $f_\pm = \max(\pm f, 0)$. $f \ge 0 \Rightarrow \fint_A f := \sup\{\oint_D f : D \subset A \text{ compact rectifiable}\}$

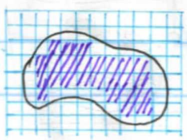
otherwise $\fint_A f := \fint_A f_+ - \fint_A f_-$

Thm 3. If $A \subset \mathbb{R}^n$ is bdd & open, $f: A \to \mathbb{R}$ is bdd cont.

~~Thm 4.~~ then $\fint_A f$ exists. If also $\oint_A f$ exists, they are equal.

pf of thm 3 (quick) $\fint \ \substack{\le \\ \ge} \ \oint_{\text{exists}}$.

Assume $\oint_A f$ exists & $f \ge 0$,



$(\le)$: $\oint_D f \le \oint_A f \Rightarrow \sup_D \text{LHS} \le \text{RHS}$. If $D \subset A$ compact & recti, $\fint_A f \le \oint_A f$

$(\ge)$: $L(f|_A, P) \overset{\text{nts}}{\le} \fint_A f$ where P is a ~~recti~~partition of a rectangle $Q \supset A$.

Assuming this, $\sup \text{LHS} \le \text{RHS}$, $\oint_A f \le \fint_A f$

In some details, $L(f|_A, P) = \sum_{R \in P} \underbrace{m_R(f|_A)}_{0 \text{ if } R \not\subset A} \cdot V(R) = \sum_{\substack{R \in P \\ R \subset A}} m_R(f) \cdot V(R) = \sum_{\substack{R \in P \\ R \subset A}} \oint_R m_R(f)$.

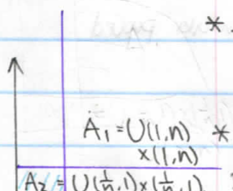
$\le \sum_{\substack{R \in P \\ R \subset A}} \oint_R f = \oint_D f \le \fint_A f$

$D := \bigcup_{\substack{R \in P \\ R \subset A}} R$ is compact & recti.

If $f = f_+ - f_-$, use linearlity. □

Thm 4. If $A \subset \mathbb{R}^k$ is open, $f: A \to \mathbb{R}$ cont, $U_1 \subset U_2 \subset U_3 \subset \cdots$ are open and $\bigcup_{n=1}^{\infty} U_n = A$, then

$\fint_A f$ exists iff $\fint_{U_n} |f|$ is bdd & then $\fint_A f = \lim_{n \to \infty} \fint_{U_n} f$.



*. Let r_n be an connection $\mathbb{Q} \cap [0,1]$. $A = \bigcup_n (r_n - \frac{0.1}{2^n}, r_n + \frac{0.1}{2^n})$ open $f: A \to \mathbb{R}$, $f(x) = 1$.

$\fint_A f$ exists, yet $D(f|_A) \supset A^c$ so it is big.

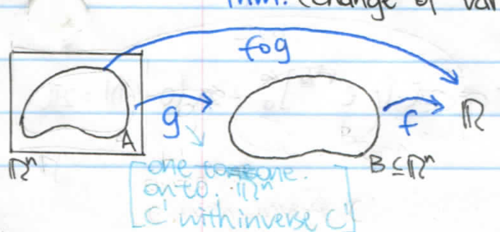
* $f(x,y) = \frac{1}{x^2 y^2}$ on. i) $A_1 = (1,\infty) \times (1,\infty)$ ii) $A_2 = (0,1) \times (0,1)$

By thm 4. $\fint_{A_1} f = \lim_{n \to \infty} \fint_{U_{1,n}} f \overset{\text{thm3}}{=} \lim_{n \to \infty} \oint_{U_{1,n}} f = \lim_{n \to \infty} \int_{[1,n] \times [1,n]} \frac{dx\,dy}{x^2 y^2} = \lim_{n \to \infty} \left(\int_1^n \frac{dx}{x^2}\right)\left(\int_1^n \frac{dy}{y^2}\right)$

$\sim \lim_{n \to \infty} \frac{n-1}{n} \cdot \frac{n-1}{n} = 1$.

$\fint_{A_2} f = \lim_{n \to \infty} \left(\int_{1/n}^1 \frac{dx}{x^2}\right)\left(\int_{1/n}^1 \frac{dy}{y^2}\right) = \lim_{n \to \infty} (n-1) \cdot (n-1)$ does not exist.

Thm. (change of var. thm, sec 17)



Let $g: A \to B$ be a diffiomorphism of open sets in $\mathbb{R}^n$ (meaning 1-1, onto, C' and inverse is C'). Let $f: B \to \mathbb{R}$ be cont. then the following two integrals each exist if the other exist, and then they are equal. $\int_B f = \int_A (f \circ g) \cdot J_g$

where $J_g: A \to ~~B~~\mathbb{R}$ is the "Jacobian of g". $J_g(x) = |\det Dg(x)|$.

Example: $[A] \xrightarrow{g} [B] \xrightarrow{f} \mathbb{R}$ $\quad J_g = |\det(g')| = |g'|$.

$\int_A f(g(x))(g'(x)) = \int_B f(y)$.

Remaindar

Dec. 1.

*Let g from A to B be a diffomorphism (1-1, onto, C', inverse is C' (equiv. to diff.)) of open sets (A, B open).

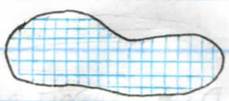
Let f be a function from $B \to \mathbb{R}$. then f is integrable on B iff $f \circ g \cdot J_g$ integrable on A

$J_g = |\det Dg(x)|$

And if they are integrable, the integrals are equal. i.e. $\int_B f = \int_A (f \circ g) \cdot J_g$.

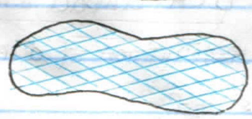
* Why true?

*. $\int_B f$



$B = \bigcup R$ (rectangles) then $\int_B f = \sum V(R_i) f(x_i)$ where $x_i \in R_i$

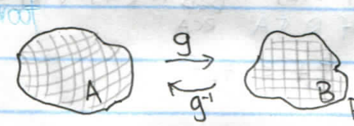
approcher as rect. goes smaller & smaller.



$B = \bigcup P_i$ (parallgram) then $\int_B f = \sum V(P_i) f(x_i)$ where $x_i \in P_i$

Proof

*



$A \underset{g^{-1}}{\overset{g}{\rightleftarrows}} B \xrightarrow{f} \mathbb{R}$. $\quad P_i \subseteq B,\ y_i \in P_i$

What is $\int_B f$?

$\int_B f = \sum_i V(P_i) f(y_i)$

take every P_i into a shape in A. which is very samilar to a paralgram.

$g^{-1}(P_i) = P_i'$ nearly parallogram. $y_i = g(x_i)$ where $x_i \in P_i'$

$\int_B f = \sum_i V(P_i) f(g(x_i)) = \sum_i V(P_i') \frac{V(P_i)}{V(P_i')} \cdot f \circ g(x_i)$

Lemma: (key) If $L: \mathbb{R}^n \to \mathbb{R}^m$ linear and P' is a paralogram in $\mathbb{R}^n$, then $V(L(P')) = |\operatorname{Det} L| \cdot V(P')$

i.e. (the determinament measures how much the volume changes)

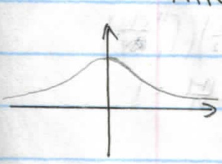
(Lemma $\Rightarrow$ thm): $\int_B f = \sum_i V(P_i') \frac{V(P_i)}{V(P_i')} \cdot f \circ g(x_i) = \sum_i V(P_i') \frac{V(g'(x_i) P_i')}{V(P_i')} \cdot f \circ g(x_i)$

$= \sum_i V(P_i') \cdot J_g(x_i) \cdot f \circ g(x_i)$ ~ formula proved.

Example Compute $\int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\, dx\, dy = G_2$

Attempt 1: Use Fubini. $G_2 = \int dx \int dy\, e^{-x^2/2} e^{-y^2/2} = \int dx\, e^{-x^2/2} \underbrace{\int dy\, e^{-y^2/2}}_{G_1}$

$= (G_1) \int dx\, e^{-x^2/2} = (G_1)^2$



where $G_1 = \int_{-\infty}^{\infty} dx\, e^{-x^2/2}$ ← the most important definite integral in math.

Attempt 2: Integral over $\mathbb{R}^2$. (use polar coordinate). $(x,y) \mapsto (r\cos\theta, r\sin\theta)$ $r > 0$.

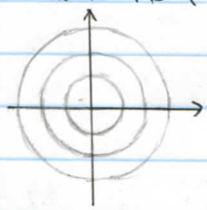
$J_g \binom{r}{\theta} = \left| \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right| = |r| = r$

therefore $G_2 = \int_{\substack{r \in [0,\infty) \\ \theta \in [0, 2\pi]}} e^{-r^2/2} \cdot r \overset{\text{Fubini}}{=} 2\pi \int_0^\infty e^{-r^2/2} \cdot r = 2\pi \cdot [-e^{-r^2/2}]_0^\infty = 2\pi(0-(-1)) = 2\pi$

$\Rightarrow G_1 = \sqrt{G_2} = \sqrt{2\pi}$

Remark: we use 2-var. integral to find $(x^x)'$

Back to the function $e^{-(x^2+y^2)/2}$, it is constant at circles. (as $(x^2+y^2) = |(x,y)|^2$. distance from $(0,0)$)



Partition f annualar. Suppose

$G_2 \approx \sum_i V(A_i) f(x_i) \quad x_i \in A_i$

$= \sum_i V(\text{circle of radius } r_i) \, e^{-r_i^2/2} \cdot r_i$

$= \int_r e^{-r^2/2} \, 2\pi r$ ← length of circle

$= 2\pi$

Define $G_N = \int_{\mathbb{R}^n} e^{-x^2/2} = (G_1)^n = (2\pi)^{n/2}$

$= \int_0^\infty dr \, e^{-r^2/2} \, r^{n-1} \, \mathrm{vol}(S_p^{n-1})$ ← use the riddle to solve this.