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Read Along: BDP Chapter 7

Systems of constant-coefficient linear homogeneous ODE

$A \in M_{n\times n}(\mathbb{R})$

$y' = Ay \quad y(0) = y_0$

What's e^{Ax}?

"sol'n": $y(x) = e^{Ax}y_0$

"$(e^{Ax})' = Ae^{Ax}$"

Definition

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = I + tA + \frac{t^2A^2}{2} + \cdots$$

Property 1

Converges!

Sketch

Suppose every entry of A is $\le M$ in size, then

$$\left| \text{an entry of } \frac{t^k A^k}{k!} \right| \le \frac{t^k n^{k-1} M^k}{k!} \xrightarrow[\text{rapidly}]{\text{very}} 0$$

A^k:

$(A)(A)(A)\cdots$

(A) | $\le n\cdot m^2$ | $n^2m^3 \le n\cdot nm^2\cdot m$ | $n^3m^4 \le nn^2m^3\cdot m$ | $\cdots$

in A^k, each entry is bounded by $n^{k-1}M^k$

Property 2

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e^{t(0)} = e^{0(-)} = I + 0 + \cdots = I$$

$e^0 = 0^0 + \cancel{\frac{1}{1!}0^1} + \cancel{\frac{1}{2}0^2} + \cdots$

$= ?$

$e^x = \sum \frac{x^k}{k!} = \underline{1} + x + \frac{x^2}{2} + \cdots$

Property 3

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}$$

$$e^{tD} = \begin{pmatrix} e^{t\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda_n} \end{pmatrix}$$

Property 4

If $A, B \in M_{n\times n}$ and $AB = BA$, then

$e^{A+B} = e^A e^B$, $e^{t(A+B)} = e^{tA} e^{tB}$

proof

$$(A+B)^2 = (A+B)(A+B)$$
$$= AA + AB + BA + BB$$
$$= A^2 + 2AB + B^2$$

$$(A+B) = \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j}$$

$$e^{A+B} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j} \qquad k = i+j$$

($k-j$ = i)

$$= \sum_{i,j=0}^{\infty} \frac{1}{(i+j)!} \binom{i+j}{j} A^j B^i \qquad \boxed{\binom{i+j}{j} = \frac{(i+j)!}{j!\,i!}}$$

$$= \sum_{i,j=0}^{\infty} \frac{1}{i!\,j!} A^j B^i$$

$$= \left(\sum_{j=0}^{\infty} \frac{1}{j!} A^j\right)\left(\sum_{i=0}^{\infty} \frac{B^i}{i!}\right)$$

$$= e^A e^B$$

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Property 5

$$e^{(t+s)A} = e^{tA}e^{sA}$$

Property 6

$$\frac{d}{dt}e^{tA} = Ae^{tA}$$
$$= e^{tA}\cdot A$$

$$\boxed{\begin{array}{l} \text{"}\frac{d}{dA}(e^{A})\text{"} \\ \exp : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n^2} \\ d(\exp) = \text{It's complicated} \end{array}}$$

proof

$$\frac{d}{dt}\left(\sum \frac{t^kA^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{k\cdot t^{k-1}A^k}{k!}$$
$$= \sum_{k=1}^{\infty} \frac{t^{k-1}A^k}{(k-1)!}$$
$$= A\sum_{m=0}^{\infty} \frac{t^mA^m}{m!} \quad , m=k-1$$
$$= Ae^{tA}$$

□

⇒ We've solved the ODE

$y(x) = e^{Ax}y_0$

$y(0) = e^{A\cdot 0}y_0 = I\cdot y_0 = y_0$ ✓

$y'(x) = \frac{d}{dx}(e^{Ax}y_0) = Ae^{Ax}y_0 = Ay$ ✓

Property 7

$$e^A = e^{C^{-1}DC} = C^{-1}e^DC \quad , \quad A = C^{-1}DC$$

proof

$$e^{C^{-1}DC} = \sum \frac{(C^{-1}DC)^k}{k!}$$

$$\boxed{\begin{array}{l} (C^{-1}DC)^k = \underbrace{C^{-1}DC\cdot C^{-1}DC\cdots\cdots C^{-1}DC}_{k \text{ times}} \\ \quad = C^{-1}D^kC \end{array}}$$

$$= \sum \frac{C^{-1}D^kC}{k!}$$
$$= C^{-1}\left(\sum \frac{D^k}{k!}\right)C$$
$$= C^{-1}e^DC$$