Thm: $K_{3,3}$ & $K_5$ are not planar.

Def: If G is a plane graph
v: vertices
e: edges
f: faces / r: regions

$\chi(G) := v - e + f$ "The Euler characteristic of G"

Thm: If G is a conn. plane graph. $\chi(G) = 2$

By induction ...

~~WLOG G has no deg 1 or 2 vertices.~~

WLOG G has no deg 1 vertices.

Now assume G has no deg 1 vertex.

Pick an edge

where did I use connectivity?

Can do it or else graph is : :

△ △

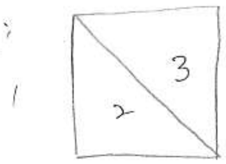
$\chi = v - e + f$
$= 6 - 6 + 3$
$= 3$

$2e = \sum_{x \in V} \deg(x)$.

If G is a plane graph and F is the set of its faces

then $2e = \sum_{y \in F} \deg(y)$ when $\deg(y) = $ # of edges around it.
$\geq 3$ if $e > 1$

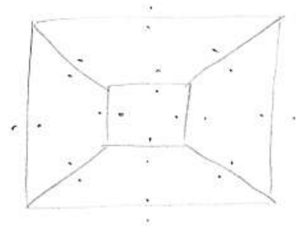
e.g.:




deg(2) = 3
deg(3) = 3
deg(1) = 4

$2 \cdot 5 = 3 + 3 + 4$ ✓

pf:



$2e = \#\text{ dots} = \sum_y \deg(y)$

$$2e = \sum_{y \in F} \deg(y) \geq 3f$$

So $\exists p \geq 0$ s.t. $2e = 3f + p$

$\Rightarrow f = \frac{2}{3}e - \frac{p}{3}$

$$\begin{aligned} \Rightarrow \quad 2 &= v - e + f \\ &= v - e + \tfrac{2}{3}e - \tfrac{p}{3} \\ &= v - \tfrac{1}{3}e - \tfrac{p}{3} \end{aligned}$$

So $6 = 3v - e - p$

$e = 3v - 6 - p$

$\leq 3v - 6$ in any plane graph or planar

In $K_5$, $e = 10$, $v = 5$,

$10 \leq 3 \cdot 5 - 6 = 9$ $(\Rightarrow\Leftarrow)$

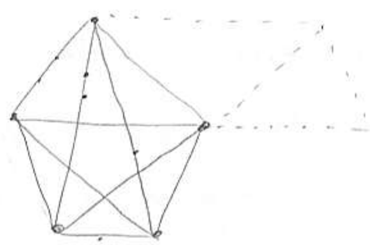
So $K_5$ is not planar.

<u>Thm</u>: (Kuratowski)

A graph $G$ is planar iff it does not contain a $K_5$ configuration or a $K_{3,3}$ configuration.

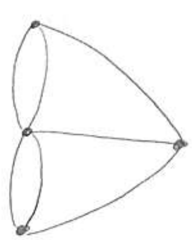
A $K_5/K_{3,3}$ configuration in $G$ is a subgraph isom. to $K_5/K_{3,3}$ which its edges possibly partitioned using bivalent vertices.

<u>e.g</u>:



<u>Königsberg</u>

Can we walk the seven bridges of Königsberg while crossing each one exactly once?



"multigraph"

A graph where multiple edge & selfedges are attained

<u>Def</u>: An Euler cycle in a multigraph $G$ is a walk on the edges of $G$ that visits every edge exactly once and return to the starting pt at the end.

Königsberg $\Leftrightarrow$ Is there an Euler cycle in  ?

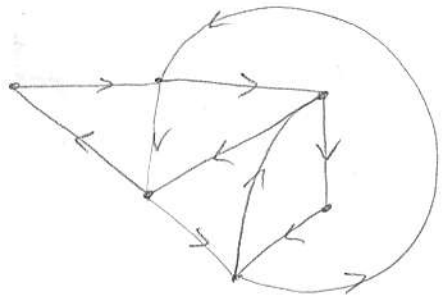
$\Leftrightarrow$ No

Thm: (Euler. 1736)

with no isolated vertices

multigraph G has an Euler cycle iff it is conn. and all vertices in it are of even degree.

pf: ($\Rightarrow$) Assume G has an Euler cycle that G is conn.



Also an Euler cycle enters every vertex the same number of times as it exits, hence degrees are even

($\Leftarrow$) Assume G is conn. $\forall x \in V.\ deg(x) \in 2\mathbb{Z}$

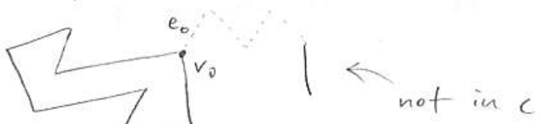
Consider a maximal simple edge-cycle in G

no repeating edges

Call it C

If C goes through all edges. we're done

Otherwise, let $v_0$ be a vertex along C that is incident to an edge $e_0$ not in C

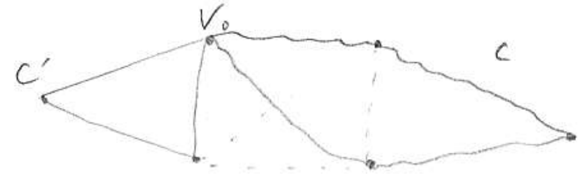



Start walking in G-C starting from $v_0$ in the direction of $e_0$. As G-C has vertices of even degree this walk terminates only when we're back to $v_0$. forming a new cycle $C'$.

now consider $C \cdot C'$

This is still a simple cycle strictly bigger than $C$

Contradicting the maximality of $C$.

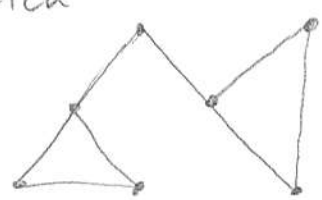



$\square$

An Euler <u>trail</u> is the same except with different start & end pts

<u>Thm</u>: $G$ has an Euler trail iff it has exactly two vertices of odd deg, in fact, these will be the start and end pts of any Euler trail.

<u>pf</u>: Sketch



($\Rightarrow$) easy