e.g $f(x)=x$. is uniformly continuous. ($\delta=\varepsilon$, not depend on x)

Thm 2. Every unif. cont. function on Q is integrable.

Thm 1. Every continuous function on a compact set is unif. cont.

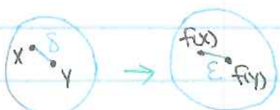
proof (thm 2). Suppose f is unif. cont. on $Q\subset\mathbb{R}^n$, let $\varepsilon>0$ be given. By unif. cont, find δ s.t. if $x,y\in Q$, $|x-y|<\delta$ then $|f(x)-f(y)|<\frac{\varepsilon}{\mathrm{vol}(Q)}$

Find a very fine partition P of Q, s.t. if $R\in P$, & $x,y\in R$, $|x-y|<\delta$,

Now for any $R\in P$ $M_R(f)-m_R(f)=\sup_R f(x)-\inf_R f(x)\le\frac{\varepsilon}{2\mathrm{vol}(Q)}$

$$U(f,P)-L(f,P)=\sum_R V(R)(M_R(f)-m_R(f))\le\sum_R V(R)\frac{\varepsilon}{2\mathrm{Vol}(Q)}=\frac{\varepsilon}{2\mathrm{vol}(Q)}\sum_R V(R)$$
$$=\frac{\varepsilon}{2\mathrm{vol}(Q)}\mathrm{Vol}(Q)=\frac{\varepsilon}{2}<\varepsilon$$

So by Riemann, f is integrable on Q. □.

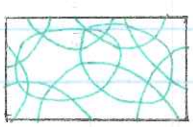


Nov. 9.

proof (thm 1)

lemma (Lebesgue number lemma) (prob C of HW2).

If $\{U_\alpha\}$ is an open cover of a compact space (X,d) then $\exists\,\delta>0$ (called Lebesgue number of $\{U_\alpha\}$) s.t. any ball $B=U(x,\delta)$ is contained in at least one of U_α.



Sketch of proof: $\Delta(x)=\sup\{r:\exists\alpha \text{ s.t. } U_\alpha\supset U(x,r)\}>0$.

$\Delta(x)$ is cont. ⇒ attains its min $=\delta>0$, & that's our number.

proof: (thm 1): for every pt $z\in X$, find an open nbd U_z s.t. $x\in U_z$, $d(f(x),f(z))<\frac{\varepsilon}{2}$ (possible because f is cont.).

if $x,y\in U_z$, then

$d(f(x),f(y))\le d(f(x),f(z))+d(f(z),f(y))<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$ ⊛

Now using the Lebesgue number lemma, find δ s.t. any ball of radius δ is contained in one of the U_z's (possible because X is compact & $\{U_z\}$ covers all z's & hence covers X). Now if $d(x,y)<\delta$ then $y\in U(x,\delta)$

So $\exists z$ s.t. $U(x,\delta)\subset U_z$ and then $x\in U(x,\delta)\subset U_z$, $y\in U(x,\delta)\subset U_z$,

So $x,y\in U_z$, so $d(f(x),f(y))<\varepsilon$ by ⊛. □.

Thm: A bdd function $f:Q\to\mathbb{R}$ (where $Q\subseteq\mathbb{R}^n$) is integrable iff the set of discontinuities of f (disco-set) is of "measure 0".

Def. Let $f:X\to Y$ be disco-set of f. $D=D(f)=\{x\in X: f \text{ is not cont. at } x\}$

Example: Suppose $A\subset X$ "the indicator function of A" $1_A(x)=\begin{cases}1 & x\in A\\ 0 & x\notin A\end{cases}$

$D(1_A)=\mathrm{Bd}(A)$

Def. A set $A \subset \mathbb{R}^n$ is "of measure 0" if for every $\varepsilon > 0$ there exists a countable collection R_i of recentages in $\mathbb{R}^n$ s.t. i) $A \subset \bigcup_i R_i$ ii). $\sum_{i=1}^{\infty} V(R_i) < \varepsilon$

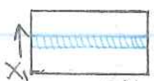
e.g. 0. $\{x\}$ is of measure 0 pf: $\boxed{\cdot}$

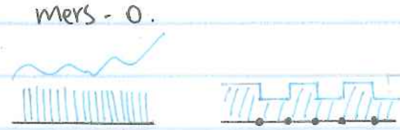
e.g. 1: $\mathbb{Q} \subseteq \mathbb{R}$ is of measure 0

Indeed, given $\varepsilon > 0$, let q_i be a listing of the elements of $\mathbb{Q}$, $\mathbb{Q} = \{q_i\}$.

take $R_i = [q_i - \frac{0.9\varepsilon}{2^{i+1}}, q_i + \frac{0.9\varepsilon}{2^{i+1}}]$ $\bigcup R_i \supset \mathbb{Q}$ $\sum V(R_i) = \sum \frac{0.9\varepsilon}{2^i} = 0.9\varepsilon < \varepsilon$

Later $[0,1] \subset \mathbb{R}$ $\longmapsto$ isn't of meas-0.

yet  $\{0\} \times [-1,1]^{n-1} \subset [-1,1]^n$ is a set of meas-0.

pf: take $R_1 = [-\delta, \delta] \times [-1,1]^{n-1}$ δ here is small. 

mathematicians think it is silly.

Nov. 11.

"Intuitionistic Logic" $\sqrt{2}^{\sqrt{2}}$ < rational $\vee$ / not-rational $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$

Imagine a machine $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0. \end{cases}$ Ask the machine. what is x. go 0.000000 ---, won't know output is 0 or not.

$\mathbb{Q} \subset \mathbb{R}$. For a rectangle Q in $\mathbb{R}$? Ⓐ. $\mathrm{Bd}\, Q$ is of meas-0. yet Ⓑ. Q is not.

Sketch of proof (of B). Assume $Q \subset \bigcup_{i=1}^{\infty} R_i$.

i) WLOG, $\mathrm{int}(R_i)$ cover Q.

pf: $\prod [c_j, d_j] \to \prod [c_i - \delta, d_i + \delta]$ (inflate each R_i by a tiny bit $\Rightarrow R_i'$).

$V(R_i') = 1.01\, V(R)$. $R_i \subset \mathrm{int}\, R_i'$

ii). WLOG $Q \subset \bigcup_{i=1}^{N} R_i$ (compactness) iii) WLOG $Q = \bigcup_{i=1}^{N} R_i$ (simple crop).

Now find a P of Q s.t. every R_i is the union of $S \in P$ $\sum_{i=1}^{N} V(R_i) \overset{?}{=} \sum_{i=1}^{N} \sum_{S \in P} V(S) \geq \sum_{S \in P} V(S) \overset{?}{=} V(Q)$.

Aside If $\{R_i\}_{i=1}^{N}$ cover $\mathbb{Q} \cap I$, then $\sum V(R_i) \geq 1$. $\bigcup R_i \supset \mathbb{Q} \cap I$.

$\overline{\bigcup R_i} \supset I$ (take closure). but $\overline{\bigcup R_i} = \bigcup R_i$ as closed.

By pervious result $\sum V(R_i) > V(I) = 1$. $\square$.

properties. 1. If A is measure-0 & $B \subset A$ is measure-0.

2. A countable union of meas-0 sets is of measure-0.

proof (of 2) If A_i is of mea-0 & $\varepsilon > 0$ is given, cover A_i with countable many R's s.t. $\sum_{j=1}^{\infty} V(R_{ij}) < \frac{\varepsilon}{2^i}$, the totally of R_i's cover $\bigcup A_i$ and

$\sum_{ij} V(R_{ij}) = \sum_i \sum_j V(R_{ij}) < \sum \frac{\varepsilon}{2^i} = \varepsilon$ $\square$.

prop. 3: Def does not change if $R_i \to \mathrm{int}\, R_i$. $A \subset \bigcup \mathrm{int}\, R_i$..

proof: Use "inflation".

Is $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ of measure 0?

$\underset{\text{of mea-0}}{\mathbb{Q}} \cup (\mathbb{R} \setminus \mathbb{Q}) = \underset{\text{not mea-0.}}{\mathbb{R}}$ if $\mathbb{R} \setminus \mathbb{Q}$ is of mea-0 then ~~by~~ $\mathbb{R}$ is of mea-0. $\Rightarrow\Leftarrow$