$$f(a+h) = f(a) + (Df)(a)\cdot h + o(h)$$

$$\Leftrightarrow \lim_{h\to 0} \frac{f(a+h)-f(a)-Df(a)\cdot h}{|h|} = 0$$

If f is diff. then

$$Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \quad \text{"Jacobian matrix"}$$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Thm: $f : \mathbb{R}^n \to \mathbb{R}$. Assume $\frac{\partial f}{\partial x_i}$ exist and are cts near a.
Then f diff at a

Lem1: For any small $h \in \mathbb{R}^n$, $\exists\, q_1, \dots, q_n$ in $B(a, |h|)$ s.t.

$$f(a+h) - f(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(q_i)\cdot h_i$$

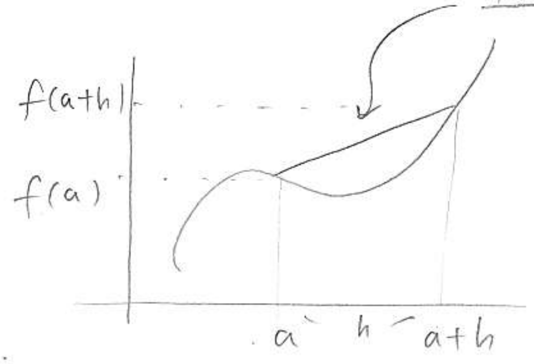
Lem2: (MVT, proven in 157)

If $g : \mathbb{R} \to \mathbb{R}$ s.t f' exists & is cts near a.

and h is "small" then $\exists\, q \in (a, a+h)$

$$f(a+h) - f(a) = f'(q)\cdot h$$

$$\frac{f(a+h)-f(a)}{h} = f'(q)$$



pf: (Lem $\Rightarrow$ Thm)

$$B = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \ldots, \frac{\partial f}{\partial x_n}(a) \right)$$

$$\frac{f(a+h) - f(a) - B \cdot h}{|h|} = \frac{\sum \frac{\partial f}{\partial x_i}(q_i) \cdot h_i - \sum \frac{\partial f}{\partial x_i}(a) h_i}{|h|}$$

$$= \frac{\sum \left(\frac{\partial f}{\partial x_i}(q_i) - \frac{\partial f}{\partial x_i}(a) \right) h_i}{|h|}$$

$$\to 0 \quad \text{as} \quad h \to 0$$

pf: (Lem)

Let $P_0 = a$

$P_1 = a + h_1 e_1$

$P_2 = P_1 + h_2 e_2 = a + h_1 e_1 + h_2 e_2$

$\vdots$

$P_n = P_{n-1} + h_n e_n = a + h_1 e_1 + \cdots + h_n e_n = a + h$

then telescopic summation

$$f(a+h) - f(a) = \sum_{i=1}^{n} \underbrace{f(P_i)}_{y_i} - \underbrace{f(P_{i-1})}_{y_i}$$

$$= \sum_{i=1}^{n} \left(f(P_{i-1} + h_i e_i) - f(P_{i-1}) \right)$$

by MVT

$\Rightarrow \exists t_i \in [0,1]$ s.t. $q_i = P_{i+1} + t_i h_i e_i$, $f(a+h) - f(a) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(q_i) \cdot h_i$ $\square$

<u>Def</u>: f is of class C^1 on $A \subset \mathbb{R}$ if $\frac{\partial f}{\partial x_i}$ exist and are cts on A.

<u>Def</u>: f is of class C^r $(r \in \mathbb{N},\ r > 1)$ if $\frac{\partial f}{\partial x_i}$ exist and are class of C^{r-1}

<u>Def</u>: A fct is C^0 if it is cts.

<u>e.g</u>: $f(x, y)$ is C^2 iff $\underset{\partial_x f}{\frac{\partial f}{\partial x}}$ and $\underset{\partial_y f}{\frac{\partial f}{\partial y}}$ are C^1

iff $\partial_x \partial_x f,\ \partial_y \partial_x f,\ \partial_x \partial_y f,\ \partial_y \partial_y f$ are C^0 $(= \text{cts})$

<u>Thm</u>: If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is C^2, then for any i, j

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

$$\partial_i \partial_j f = \partial_j \partial_i f$$