Nov 19, 2012

Theorem

π is irrational

proof

Suppose $\pi = \frac{a}{b}$

Consider $f_n(x) = \frac{x^n(a-bx)^n}{n!} = \frac{x^n b^n(\pi - x)^n}{n!}$

* All coefficients of numerator are integers $\Rightarrow f^{(k)}(0) \in \mathbb{Z}$

* $f(x) = f(\pi - x)$ $\Rightarrow f^{(k)}(\pi) \in \mathbb{Z}$

Consider for large n $\quad 0 < \int_0^\pi f_n(x) \sin x \, dx < 1$

$\downarrow$ by repeated integration by parts

$0 +$ boundary terms $\rightarrow$ (evaluation of sin or cos at 0 or π) $\cdot$ (derivatives of f and 0 at π) $\in \mathbb{Z}$

Contradiction!

Theorem 1

Given $\sum_{n=0}^{\infty} a_n x^n$ $\exists R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$

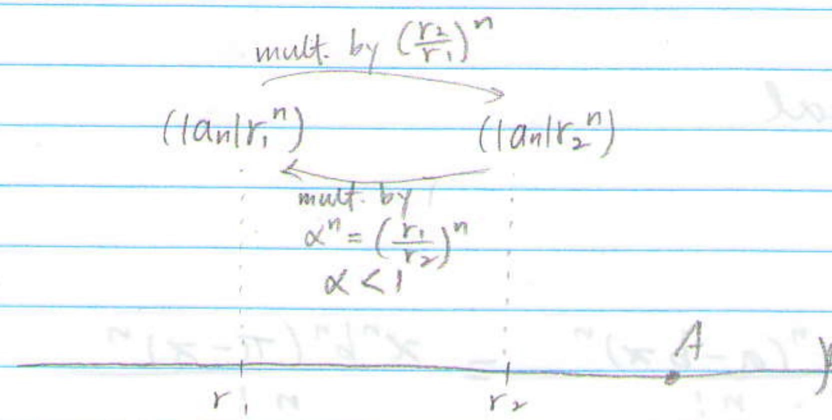
"the radius of convergence" s.t. $\sum a_n x^n$ absolutely converges if $|x| < R$ & diverges if $|x| > R$

(if $|x| = R$, "it depends")

proof

Given a sequence b_n, if $\alpha < 1$

$$\underset{A}{\begin{pmatrix} b_n \text{ summable} \\ \sum b_n \text{ converges} \end{pmatrix}} \Rightarrow \underset{B}{(b_n \to 0)} \Rightarrow \underset{C}{\begin{pmatrix} b_n \text{ is} \\ \text{bounded} \end{pmatrix}} \Rightarrow \begin{pmatrix} \alpha^n b_n \\ \text{summable} \end{pmatrix}$$



$R = \sup(r : A \text{ holds})$
$= \sup(r : B \text{ holds})$
$= \sup(r : C \text{ holds})$

$\Rightarrow$ If A or B or C holds for $(|a_n|r_2^n)$
then A, B & C hold for $(|a_n|r_1^n)$
if A or B or C hold somewhere, then A & B & C hold everywhere to the left

<u>Theorem 2 (loose)</u>

1. If f has a formula, then it has a natural then it has a natural extension to $\mathbb{C}$
2. In that case, R, the radius of convergence of f's Taylor series $\sum a_n x^n$, $a_n = \left(\frac{d^n}{dx^n} f\right)/n!$ is the distance from 0 to the nearest point in $\mathbb{C}$ where the formula for f genuinely fails.

<u>examples</u>

1. $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ makes sense in $\mathbb{C}$ everywhere so $R = \infty$
2. $f = \dfrac{1}{1+x^2}$ makes sense for every $x \in \mathbb{R}$ has an extension to $\mathbb{C}$ has divergence by 0 at $x = \pm i$
so $R = d(0, \pm i) = 1$
$f = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$ & indeed $R = 1$

3. C_n = Catalan numbers: 1, 1, 2, 5, 14, 42, ...

$$\sum C_n x^n = \frac{1-\sqrt{1-4x}}{2x} \cdot \frac{1+\sqrt{1-4x}}{1+\sqrt{1-4x}}$$

$$= \frac{1^2 - \sqrt{1-4x}^2}{2x(1+\sqrt{1-4x})}$$

$$= \frac{4x}{2x(1+\sqrt{1-4x})}$$

$$= \frac{2}{1+\sqrt{1-4x}}$$

First problem at $x = \frac{1}{4}$

$\Rightarrow R = \frac{1}{4}$.

C_n grow faster than 3.99^n & slower than 4.01^n

Indeed $\sum C_n \frac{1}{4.01^n}$ converges

$\Rightarrow C_n/4.01^n \longrightarrow 0$

Problem

$y'' + y = 0$

Given $y'' + p(x)y' + q(x)y = g(x)$, find & study a power $y = \sum a_n x^n$ that solves this equation.

Airy's equation $y'' = xy$

sub. $y = \sum_{n=0} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$\rightsquigarrow \quad (n+2)(n+1)a_{n+2} = a_{n-1} \quad$ for $n \geq 1$

$y(0) = a_0 \qquad y'(0) = a_1 \qquad a_2 = 0$

$a_3 = (3 \cdot 2)^{-1} a_0 = \dfrac{a_0}{2 \cdot 3}$

$a_4 = \quad \cdots \quad = \dfrac{a_1}{3 \cdot 4}$

$a_5 = \quad \cdots \quad = \dfrac{a_2}{\cdots} = 0$