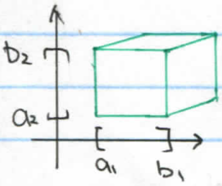
$0 = f(x, g(x))$ $\quad \mathbb{R}^n_x \xrightarrow[\begin{pmatrix}\partial x/\partial x \\ \partial g/\partial x\end{pmatrix}]{x \mapsto \binom{x}{g(x)}} \mathbb{R}^n \times \mathbb{R}^k \xrightarrow[\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{pmatrix}]{\binom{x}{y} \mapsto f(x,y)} \mathbb{R}_k$

$\downarrow$ taking D

$0 = D[\ldots]$

$= \begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{pmatrix}\begin{pmatrix} I \\ Dg\end{pmatrix} \quad \Rightarrow Dg = -\underbrace{\left(\frac{\partial f}{\partial y}\right)^{-1}}_{k\times k}\underbrace{\left(\frac{\partial f}{\partial x}(x, g(x))\right)}_{k\times n}$ $\quad k\times n$

$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} Dg$ $\quad \square$



Aim of courses: stoke's thm $\int_M d\omega = \int_{\partial M} \omega$.

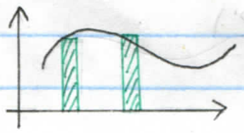
Test: Given $f: \mathbb{R}^n \to \mathbb{R}$ & $Q = \prod_{j=1}^n [a_j, b_j]$ Define $\int_Q f$.

TBC Next week.

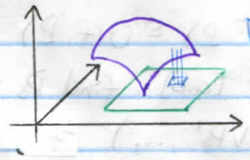
Oct 31

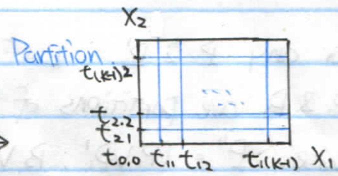
$f: \mathbb{R} \to \mathbb{R}$ $\qquad\qquad$ $f: \mathbb{R}^n \to \mathbb{R}$

$\int_a^b f(x)dx$ $\quad \int_{[a,b]} f(a,b)$. $\quad Q = \prod_{j=1}^n [a_j, b_j] = \{x \in \mathbb{R}^n : \forall j, a_j \le x_j \le b_j\}$



$\int_Q f$.





Partition $a\ t_1\ t_2 \ldots t_{k-1}\ b$

$\{t_0, \ldots t_k\}$ $a = t_0 < t_1 \cdots < t_k = b$

$P \sim \{[t_0,t_1], [t_1,t_2] \cdots [t_{k-1}, t_k]\}$. $\qquad P = (P_1, \ldots, P_n)$ P_j is a partition of $[a_j, b_j]$

$J \in P$ $m_J(f) = \inf\{f(x) : x \in J\}$ $\qquad R \in P \Leftrightarrow R = \prod_{j=1}^n [c_j, d_j]$ s.t. $\forall j, [c_j, d_j] \in P_j$

Def. f is integrable on $[a,b]$ if $\underline{\int}_{[a,b]} f = \overline{\int}_{[a,b]} f$. In that case set $\int_{[a,b]} f = \underline{\int} = \overline{\int}$

$M_J(f) = \sup\{f(x) : x \in J\}$. $\qquad m_R(f) = \inf\{f(x) : x \in R\}$ $\quad M_R(f) = \sup\{f(x) : x \in R\}$

$l(J) = l([t_{i-1}, t_i]) = t_i - t_{i-1}$ $\qquad V(R) = V(\prod [c_j, d_j]) := \prod_{j=1}^n (d_j - c_j)$

$\Rightarrow L(f, P) = \sum_{J \in P} l(J) m_J(f)$ $\qquad \Rightarrow L(f,P) = \sum_{R \in P} V(R) \cdot m_R(f)$.

$U(f, P) = \sum_{J \in P} l(J) M_J(f)$ $\qquad U(f,P) = \sum_{R \in P} V(R) \cdot M_R(f)$

$\underline{\int}_{[a,b]} f = \sup_P L(f,P)$ $\quad \inf_P U(f,P) = \overline{\int}_{[a,b]} U(f,P)$. $\quad \underline{\int}_Q f = \sup_P L(f, P)$ $\quad \overline{\int}_Q f = \inf_P U(f, P)$. Def: ...

$*$ $U(f, P_1) \ge L(f, P_2)$ $\forall P_1, P_2$ partition.

Assuming above is true, $\int_Q 7$ take $P = (t_{10} = a_1 < t_{11} = b_1 ; t_{20} = a_2 < t_{21} = b_2) \sim \{Q\}$.

$m_Q(f) = 7 = M_Q(f)$. $\Rightarrow L(f, P) = V(Q) \cdot 7$ $\quad U(f, P) = V(Q) \cdot 7$ $\Rightarrow \overline{\int} = \underline{\int} = V(Q) \cdot 7$.

$\Rightarrow \int_Q 7 = 7 V(Q)$ (if above correct).

Nov. 2.

Def. $P' = (a < t_0 < t_1' < \cdots t_{k'}' = b)$. is a "refinement" of $P = (a = t_0 < t_1 < \cdots < t_k = b)$

if $\forall j, t_j \in P'$ ($\forall j \exists j', t_j = t'_{j'}$), $P' = (P_1, \cdots P_m)$ if $\forall j$ P_j' is a refinement P_j.

Lemma: If P' refines P then ① $L(f,P') \geq L(f,P)$ ② $U(f,P') \leq U(f,P)$

proof. ② May assume that P' is obtained from P by adding a singule point s on coord $\# j_0$ between $t_{j_0,i-1}$ & $t_{j_0,i}$ so $P_j = P'_j$ for $j \neq j_0$ & $P'_{j_0} = (a_j = t_{j_0,0} < t_{j_0,1} < \cdots < t_{j_0,i-1} < s < t_{j_0,i} \cdots < t = b)$

Enough to consider rectangles R' in $P = (P_1 \ldots P_n)$ of the form: ~~$R' = \prod_{j=1}^{j_0-1} J_j \times [t_{j_0,i-1},$~~

$$R' = \prod_{j=1}^{j_0-1} J_j \times [t_{j_0,i-1}, t_{j_0,i}] \times \prod_{j=j_0+1}^{n} J_j \quad \forall j \neq j_0 \quad J_j \in P_j.$$

After refinement, namely, in P', $R' \to R_1 \cup R_2$ where

$$R_1 = \prod_{j=1}^{j_0-1} J_j \times [t_{j_0,i-1}, s] \times \prod_{j=j_0+1}^{n} J_j \qquad R_2 = \prod_{j=1}^{j_0-1} J_j \times [s, t_{j_0,i}] \times \prod_{j=j_0+1}^{n} J_j$$

$$V(R') = V(R_1) + V(R_2)$$

$$\underbrace{\left(\prod_{j \neq j_0} |J_j|\right)}_{A} (t_{j_0,i} - t_{j_0,i-1}) = A(s - t_{j_0,i-1}) + A(t_{j_0,i} - s) \quad \text{then distri. law.}$$

②. $M_{R'}(f) \geq M_{R_1}(f)$. $M_{R'}(f) \geq M_{R_2}(f)$ obvious as $R' \supset R_{1,2}$

$\Rightarrow M_{R'}(f) \cdot V(R') = M_{R'}(f)(V(R_1) + V(R_2)) \geq M_{R_1}(f) V(R_1) + M_{R_2}(f) V(R_2)$

$\Rightarrow U(f,P) \geq U(f,P')$ likewise $L(f,P) \leq L(f,P')$

Lemma: For any P & P' $L(f,P) \leq U(f,P')$

proof: If P_1 & P_2 are partitions of $[a,b]$, find P_3 that refines both. $P_3 = P_1 \vee P_2$.

Likewise $P'' = (P_1 \vee P_1', P_2 \vee P_2', \ldots)$ P'' refines both P & P'

Now. $L(f,P) \leq L(f,P'') \leq U(f,P'') \leq U(f,P')$ □.

Cor. $\underline{\int}_Q f$ & $\overline{\int}_Q f$ make sence & $\underline{\int} f \leq \overline{\int} f$ □

Prop. (The Riemann condi.) f is integrable iff $\forall \varepsilon > 0$ $\exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$ □

Nov. 4

Thm. Every cont. function on Q is integrable.

Example. $f(x) = c$ is integrable on Q $\int_Q f = c \cdot V(Q)$

proof. Take $P = (a_1 = t_0 < t_n = b_1 ; a_2 < b_2 ; \cdots ; a_n < b_n)$.

$P = \{Q\}$ $L(f,P) = c \cdot V(Q)$ $U(f,P) = c \cdot V(Q)$ $U - L = 0 < \varepsilon$. $\Rightarrow f$ is integrable.

Eg. 2. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ $\quad m_R(f) = 0$. Some prev. example w/c $= 0$. $\quad m_R(f) = 1$ Some prev. example w/c $= 1$.

$a < b$. $Q = [a,b]$. $L(f,P) = 0$. $U(f,P) = 1 \cdot (b-a) = b-a$.

$U(f,P) - L(f,P) = b - a \not< \varepsilon$ f is not integrable.

Def. f cont. on Q: $\forall x \in Q$, $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall y \in Q$. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

f uniformly continuous on Q: $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x, y \in Q$ $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

example. $f(x) = x^2$ on $\mathbb{R}$. ← cont. but not unif. cont.

can find different δ for different x in cont. i.e. δ_x. Single δ in unif. cont. i.e. δ for all x

e.g $f(x)=x$. is uniformly continuous. ($\delta=\varepsilon$, not depend on x)

Thm 2. Every unif. cont. function on Q is integrable.

Thm 1. Every continuous function on a compact set is unif. cont.

proof (thm 2). Suppose f is unif. cont. on $Q \subset \mathbb{R}^n$, let $\varepsilon > 0$ be given. By unif. cont, find δ s.t. if $x, y \in Q$, $|x-y| < \delta$ then $|f(x)-f(y)| < \frac{\varepsilon}{vol(Q)}$

Find a very fine partition P of Q, s.t. if $R \in P$, & $x, y \in R$, $|x-y| < \delta$,

Now for any $R \in P$ $M_R(f) - m_R(f) = \sup_R f(x) - \inf_R f(x) \leq \frac{\varepsilon}{2vol(Q)}$

$U(f,P) - L(f,P) = \sum_R V(R)(M_R(f) - m_R(f)) \leq \sum_R V(R) \frac{\varepsilon}{2vol(Q)} = \frac{\varepsilon}{2vol(Q)} \sum_R V(R)$

$= \frac{\varepsilon}{2vol(Q)} Vol(Q) = \frac{\varepsilon}{2} < \varepsilon$

So by Riemann, f is integrable on Q. $\square$.