Nov 23. 2012

Topic 1 $xy' + p(x)y = 0$, p analytic, $P(x) = \sum_{n=0}^{\infty} P_n x^n$

"has a convergent power series expansion"

why?

What if $ry' + py = 0$ $r(0) = 0$

$$xy' + \frac{xp}{r} y = 0$$

We're studying $ry' + py = 0$ where $\frac{xp}{r}$ has a p.s. expansion.

How?

First try $xy' + P_0 y = 0$

$\rightsquigarrow$ solution is $y = x^{-P_0} = x^{\alpha}$ $\alpha = -P_0$

So for $xy' + py = 0$ try $y = x \sum a_n x^n$

$$y = \sum_{n=0}^{\infty} a_n x^{n+\alpha} \quad [\text{take } a_0 = 1]$$

Compare coefficients of $x^{n+\alpha}$

$$a_n(n+\alpha) + \sum_{j=0}^{n} P_j a_{n-j} = 0$$

$$(n+\alpha+P_0)a_n = -\sum_{j=1}^{n} P_j a_{n-j}$$

if $n=0$, $(n+\alpha+P_0) = 0 \longrightarrow \alpha = -P_0$

if $n>0$, $a_n = \frac{-1}{n} \sum_{j=1}^{n} P_j a_{n-j}$

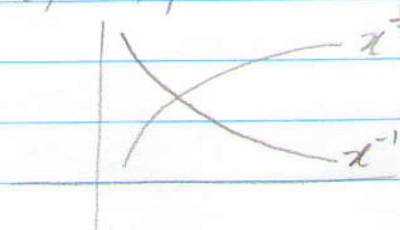
Fuch's Theorem still holds with the same conclusion

$y(x) = x^{\alpha}$ (a convergent p.s.)

Qualitative Behaviour

near $x=0$, $y \sim x^{\alpha}$

$xy' + py = 0$



example

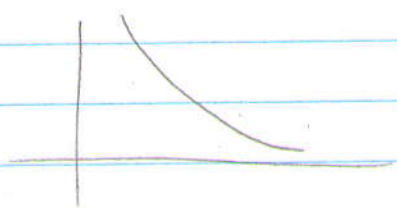
What does y look like near $x=0$, if

$$y' + \frac{e^x + e^{-x}}{x} y = 0$$

$$\rightsquigarrow xy' + (e^x + e^{-x}) y = 0$$

$$xy' + (2 + x^2 + \cdots) y$$

$$y \sim x^\alpha = x^{-2}$$



Topic 2

$$xy' + py = 0 \longrightarrow \boxed{x^2y'' + xpy' + qy = 0}$$

Why not study $xy'' + py' + qy = 0$?

$$x^2y'' + xpy' + \underbrace{xq}_{\tilde{q}} y = 0$$

Why not study $x^2y'' + py' + qy = 0$?

Pretend to solve by p.s. write recursion $a_n = n$ (mess)

so p.s. never converges

What about $ry'' + py' + qy = 0$

$$x^2y'' + x\underbrace{\left(\frac{xp}{r}\right)}_{\tilde{p}}y' + \underbrace{\frac{qx^2}{r}}_{\tilde{q}} y = 0$$

"has a regular singular point at $x=0$" if $\frac{xp}{r}$ & $\frac{x^2q}{r}$ are analytic near 0

How

Try x^α in $\underbrace{x^2y'' + xp_0y' + q_0y}_{L_0 y} = 0$

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$L_0 x^\alpha = F(\alpha) x^\alpha$ where $\underbrace{F(\alpha) = \alpha(\alpha-1) + P_0\alpha + q_0}_{\text{indicial polynomial}}$

$F(\alpha) = 0 \rightsquigarrow \alpha = \alpha_1, \alpha_2$

Now try in $x^2y'' + xpy' + qy = 0$, $y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$, $a_0 = 1$

$$F(\alpha+n)a_n = -\sum_{k=0}^{n-1} a_n[(\alpha+k)P_{n-k} + q_{n-k}]$$

$$a_n = \frac{-1}{F(\alpha+n)}(\dots)$$

Easiest case. $\alpha_1 \neq \alpha_2$, $\alpha_1 - \alpha_2 \notin \mathbb{Z}$ get

$$y_1 = x^{\alpha_1}\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)$$

$$y_2 = x^{\alpha_2}\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)$$

Fuch's Theorem still holds so all p.s. converge.

<u>example</u>

What do solutions of $y'' + \left(\frac{1}{2x^2} + \frac{1}{2(1-x^2)}\right)y = 0$ look like near $x = 0$

$$F(\alpha) = \alpha(\alpha-1) + P_0\alpha + q_0$$
$$= \alpha(\alpha-1) + \frac{1}{2}$$
$$= 0$$

$$\alpha = \frac{1}{2} \pm \frac{1}{2}i$$