$$\begin{tikzcd}
\Omega^0\mathbb{R}^3 \arrow[r, "d"] \arrow[d, leftrightarrow] & \Omega^1 \arrow[r, "d"] \arrow[d, leftrightarrow] & \Omega^2 \arrow[r, "d"] \arrow[d, leftrightarrow] & \Omega^3 \arrow[d, leftrightarrow] \\
f \arrow[r, "\text{grad}", "\nabla"'] & \text{v.f} \arrow[r, "\text{curl}", "\nabla\times"'] & \text{v.f} \arrow[r, "\text{div}", "\nabla"'] & f
\end{tikzcd}$$

Thm: $\exists!$ linear $d : \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R})$ s.t.

1. $f \in \Omega^0 \Rightarrow (df)(\xi) = D_\xi f$

   so $df = \sum \frac{\partial f}{\partial x_i} dx_i$

2. $d(w \wedge \eta) = (dw) \wedge \eta + (-1)^{\deg w} w \wedge d\eta$

3. $d^2 = 0$

pf: 1 - 3 imply

$$d(\underbrace{\sum a_I dx_I}_{w}) = \sum_{j=1}^{n} \sum_{I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I$$

$$\overset{\text{loosely}}{=} \sum_{j=1}^{n} dx_j \wedge \frac{\partial w}{\partial x_j}$$

pf of existence

Set $dw = \sum dx_j \wedge \frac{\partial w}{\partial x_j}$ verify 1-3

1. $f \in \Omega^0$

$$df = \sum dx_j \frac{\partial f}{\partial x_j} = \sum_j \frac{\partial f}{\partial x_j} dx_j$$

· to compute with $df(\xi) = D_\xi f$ eval. both sides on $\xi = (x, e_j)$

2. $d(w \wedge \eta) = d(a\,dx_I \wedge b\,dx_J)$ by linearity $w = a\,dx_I$, $\eta = b\,dx_J$

$$= d(a\,b\,dx_I \wedge dx_J)$$

$$= \sum_{j=1}^{n} dx_j \wedge \frac{\partial (ab)}{\partial x_j} dx_I \wedge dx_J$$

$$= \sum \left(\frac{\partial a}{\partial x_j} b + a \frac{\partial b}{\partial x_j} \right) dx_j \wedge dx_I \wedge dx_J$$

$$= \sum dx_j \wedge \frac{\partial a}{\partial x_j} dx_I\, b\,dx_J + (-1)^{\deg w} \sum a\,dx_I\, dx_j \frac{\partial b}{\partial x_j} dx_J$$

$$= (dw) \wedge \eta + (-1)^{\deg w} w \wedge d\eta.$$

3. $d(dw) = d\left(\sum_j dx_j \wedge \frac{\partial w}{\partial x_j} \right)$

$$= \sum_k dx_k \wedge \frac{\partial \left(\sum_j dx_j \wedge \frac{\partial w}{\partial x_j} \right)}{\partial x_k}$$

$$= \sum_k \sum_j dx_k \wedge dx_j \wedge \frac{\partial}{\partial x_k} \frac{\partial w}{\partial x_j}$$

$$\underset{j \leftrightarrow k}{=} \sum_j \sum_k dx_j \wedge dx_k \frac{\partial^2 w}{\partial x_j \partial x_k}$$

$$= -\sum_k \sum_j dx_k \wedge dx_j \frac{\partial^2 w}{\partial x_k \partial x_j}$$

$d(dw) = -d(dw) \implies d(dw) = 0$

$\square$

$\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\Omega^k(\mathbb{R}^n) \xleftarrow[\text{pullback}]{\phi^*} \Omega^k(\mathbb{R}^m) \ni w$$

$$\phi^*(w)(\xi_1, \dots, \xi_k) = w(\phi_* \xi_1, \dots \phi_* \xi_k)$$

$\xi_j \in T_x \mathbb{R}^n$

Properties: 1. ϕ^* is linear

2. $\phi^*(w \wedge \eta) = (\phi^* w) \wedge (\phi^* \eta)$

3. $(\phi \circ \psi)^* = \psi^* \circ \phi^*$

$\Bigg\}$ Done Before

4. $\phi^*(dw) = d(\phi^* w)$ — TBD

$$\mathbb{R}^2_{r,\theta} \xrightarrow{\phi} \mathbb{R}^2_{x,y}$$

$$\phi\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$$w = \frac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2_{x,y})$$

$$d(\phi^* w) = \phi^*\left(\frac{x\,dy}{x^2+y^2}\right) + \phi^*\left(\frac{-y\,dx}{x^2+y^2}\right)$$

$$= \phi^*\left(\frac{x}{x^2+y^2}\right) \cdot \phi^*(dy) + \cdots$$

$$= \phi^*\left(\frac{x}{x^2+y^2}\right) \cdot d(\phi^* y) + \cdots$$

$$= \frac{r\cos\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\, d(r\sin\theta) + \cdots$$

$$= \frac{\cos\theta}{r}(\sin\theta\,dr + r\cos\theta\,d\theta) + \frac{-r\sin\theta}{r^2}(\cos\theta\,dr - r\sin\theta\,d\theta)$$

$$= d\theta$$