MATH 240 – FALL 2014

HOMEWORK ASSIGNMENT #1

CORRECTION

Algebra I

21/09/2014

UNIVERSITY OF TORONTO

1. **Suppose a and b are nonzero elements of a field F. Using only the field axioms, prove that $a^{-1} \cdot b^{-1}$ is a multiplicative inverse of ab. State which axioms are used in your proof.**

Let $a, b \in F$ such that $a \neq 0 \wedge b \neq 0$. First, we have $a, b \in F \Longrightarrow a \cdot b \in F$ because a field is closed under multiplication. Moreover, $a \neq 0 \wedge b \neq 0 \Longrightarrow a \cdot b \neq 0$. Therefore, since $a \cdot b$ is a nonzero element of a field, there exists a multiplicative inverse of $a \cdot b$ belonging to this field by axiom F_4 of a field. Let denote by 1 the identity element of multiplication of this field. Then:

$(a \cdot b) \cdot (a^{-1} \cdot b^{-1})$	$= (a \cdot b) \cdot (b^{-1} \cdot a^{-1})$	*by F_1: commutative property of multiplication*
	$= a \cdot (b \cdot b^{-1}) \cdot a^{-1}$	*by F_2: associativity of multiplication*
	$= a \cdot (1) \cdot a^{-1}$	*by F_4: definition of multiplicative inverse*
	$= a \cdot (1 \cdot a^{-1})$	*by F_2: associativity of multiplication*
	$= a \cdot a^{-1}$	*by F_3: definition of identity elements*
	$= 1$	*by F_4: definition of multiplicative inverse*

Finally, since we have $(a \cdot b) \cdot (a^{-1} \cdot b^{-1}) = 1$, $a^{-1} \cdot b^{-1}$ is a multiplicative inverse of $a \cdot b$.

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2. **Prove that if a and b are elements of a field F, then $a^2 = b^2$ if and only if $a = b$ or $a = -b$.**

First, we know that:

$$a^2 = b^2 \quad \Leftrightarrow a^2 - b^2 = 0 \tag{1}$$

Moreover:

$(a - b) \cdot (a + b)$	$= a \cdot (a - b) + b \cdot (a - b)$	*by F_5: distributive law*
	$= a \cdot a + a \cdot (-b) + b \cdot a + b \cdot (-b)$	*by F_5: distributive law*
	$= a^2 + a \cdot (-b) + a \cdot b + b \cdot (-b)$	*by F_1: commutative property of multiplication*
	$= a^2 + a \cdot (b - b) + b \cdot (-b)$	*by F_5: distributive law*
	$= a^2 + a \cdot 0 + b \cdot (-b)$	*by F_4: definition of additive inverse*
	$= a^2 + 0 + b \cdot (-b)$	*Demonstrated in class:* $\forall a \in F, 0 \cdot a = 0$
	$= a^2 + b \cdot (-b)$	*by F_3: definition of identity elements*

So we have:

$$(a - b) \cdot (a + b) = a^2 + b \cdot (-b) \tag{2}$$

We also have:

$b^2 + b \cdot (-b)$	$= b \cdot b + b \cdot (-b)$	
	$= b \cdot (b - b)$	*by F_5: distributive law*
	$= b \cdot 0$	*by F_4: definition of additive inverse*
	$= 0$	*Demonstrated in class:* $\forall a \in F, 0 \cdot a = 0$

So $b \cdot (-b)$ is an additive inverse of b^2. In addition, $-b^2$ is also an additive inverse of b^2. As we demonstrated in class, the inverse for addition is unique. Therefore we have: $-b^2 = b \cdot (-b)$. Finally, equation (2) become:

$$(a - b) \cdot (a + b) = a^2 - b^2 \tag{3}$$

By combining (3) and (1) we get:

$$(a - b) \cdot (a + b) = 0$$

As demonstrated in class (12th property of the theorem on basic properties of fields), we have:
$\forall c, c' \in F,\ c \cdot c' = 0 \Leftrightarrow c = 0 \vee c' = 0$

Therefore:

$$\begin{aligned}(a-b)\cdot(a+b) = 0 \quad &\Leftrightarrow (a-b) = 0 \vee (a+b) = 0\\ &\Leftrightarrow a = b \vee a = -b\end{aligned}$$

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3. **Let $F_4 = \{0, 1, a, b\}$ be a field containing 4 elements. Assume that $1 + 1 = 0$. Prove that $b = a^{-1} = a^2 = a + 1$.**

First, let us prove that:

- $\boldsymbol{b = a^{-1}}$

Since F_4 is a field, $a \cdot b$ exists and belongs to F_4 because a field is closed under multiplication. Therefore, there are 4 possible values for $a \cdot b$: $0, 1, a$ or b.

- Suppose that $a \cdot b = 0$. Then $a \cdot b = 0 \Leftrightarrow a = 0 \vee b = 0 \Rightarrow F_4 = \{0,1,b\} \vee F_4 = \{0,1,a\}$. This contradicts the fact that F_4 is a field containing 4 elements. Hence $a \cdot b \neq 0$.
- Suppose that $a \cdot b = a$. Then $a \cdot b = a \Rightarrow b = 1$ by cancellation $\Rightarrow F_4 = \{0,1,a\}$. This contradicts the fact that F_4 is a field containing 4 elements. Hence $a \cdot b \neq a$.
- Suppose that $a \cdot b = b$. Then $a \cdot b = b \Rightarrow a = 1$ by cancellation $\Rightarrow F_4 = \{0,1,b\}$. This contradicts the fact that F_4 is a field containing 4 elements. Hence $a \cdot b \neq b$.

Finally, the only possible value for $a \cdot b$ is 1. Hence:

$$a \cdot b = 1 \Leftrightarrow b = a^{-1}$$

- $\boldsymbol{b = a^2}$

Since F_4 is a field, $a^2 = a \cdot a$ exists and belongs to F_4 because a field is closed under multiplication. Therefore, there are 4 possible values for a^2: $0, 1, a$ or b.

- Suppose that $a^2 = 0$. Then $a \cdot a = 0 \Leftrightarrow a = 0 \Rightarrow F_4 = \{0,1,b\}$. This contradicts the fact that F_4 is a field containing 4 elements. Hence $a^2 \neq 0$.
- Suppose that $a^2 = 1$. Then $a \cdot a = 1 \Rightarrow a = a^{-1} \Rightarrow a = b$ because the multiplicative inverse is unique $\Rightarrow F_4 = \{0,1,a\}$. This contradicts the fact that F_4 is a field containing 4 elements. Hence $a^2 \neq 1$.
- Suppose that $a^2 = a$. Then $a \cdot a = a \Rightarrow a = 1$ by cancellation $\Rightarrow F_4 = \{0,1,b\}$. This contradicts the fact that F_4 is a field containing 4 elements. Hence $a^2 \neq a$.

Finally, the only possible value for a^2 is b. Hence:

$$b = a^{-1} = a^2$$

- $\boldsymbol{b = a + 1}$

Since F_4 is a field, $a + 1$ exists and belongs to F_4 because a field is closed under addition. Therefore, there are 4 possible values for $a + 1$: $0, 1, a$ or b.

- Suppose that $a + 1 = 0$. Then $a + 1 = 0 \Rightarrow a = 1$ because $1 + 1 = 0$ and because the additive inverse is unique $\Rightarrow F_4 = \{0,1,b\}$. This contradicts the fact that F_4 is a field containing 4 elements. Hence $a + 1 \neq 0$.
- Suppose that $a + 1 = 1$. Then $a + 1 = 1 \Rightarrow a = 0$ by cancellation $\Rightarrow F_4 = \{0,1,b\}$. This contradicts the fact that F_4 is a field containing 4 elements. Hence $a + 1 \neq 1$.
- Suppose that $a + 1 = a$. Then $a + 1 = a \Rightarrow 1 = 0$ by cancellation. This contradicts the fact that in a field the identity element for addition ("0") is different from the additive element from multiplication ("1"). Hence $a + 1 \neq a$.

Finally, the only possible value for $a+1$ is b. Hence:

$$b = a^{-1} = a^2 = a + 1$$

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4. Write the following complex numbers in the for $a + ib$, with $a, b \in \mathbb{R}$:

4.1. $\frac{1}{2i} + \frac{-2i}{5-i}$

$$\begin{aligned}
\frac{1}{2i} + \frac{-2i}{5-i} &= \frac{1}{2i} \cdot 1 + \frac{-2i}{5-i} \cdot 1 \\
&= \frac{1}{2i} \cdot \frac{5-i}{5-i} + \frac{-2i}{5-i} \cdot \frac{2i}{2i} \\
&= \frac{5-i}{10i - 2i^2} + \frac{-4i^2}{10i - 2i^2} \\
&= \frac{5-i}{10i+2} + \frac{4}{10i+2} \\
&= \frac{9-i}{10i+2} \\
&= \frac{9-i}{10i+2} \cdot 1 \\
&= \frac{9-i}{10i+2} \cdot \frac{10i-2}{10i-2} \\
&= \frac{90i - 10i^2 - 18 + 2i}{100i^2 - 4} \\
&= \frac{92i + 10 - 18}{-104} \\
&= \frac{92i - 8}{-104} \\
&= \frac{8 - 92i}{104} \\
&= \frac{8}{104} + \left(-\frac{92}{104}\right) i \\
&= \frac{1}{13} + \left(-\frac{23}{26}\right) i
\end{aligned}$$

Finally, we have $z = \frac{1}{2i} + \frac{-2i}{5-i} = \frac{1}{13} + \left(-\frac{23}{26}\right) i$ with $a = \frac{1}{13}$, $b = -\frac{23}{26}$ and $a, b \in \mathbb{R}$.

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4.2. $(1+i)^5$

Notice that: $1 + i \in \mathbb{C} \Leftrightarrow (1,1) \in \mathbb{C}$. Moreover, we have demonstrated during class that:

$$(a,b) \cdot (c,d) = (a \cdot c - b \cdot d, b \cdot c + a \cdot d)$$

Thus:

$$\begin{aligned}
(1,1)^5 &= ((1,1)^2 \cdot (1,1)^2) \cdot (1,1) && \textit{By associativity of multiplication} \\
&= \big((0,2) \cdot (0,2)\big) \cdot (1,1) \\
&= (-4,0) \cdot (1,1) \\
&= (-4,-4)
\end{aligned}$$

Finally, we have $z = (1+i)^5 = -4 - 4i$ with $a = -4$, $b = -4$ and $a, b \in \mathbb{R}$.

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5. **Fields:**
   5.1. **Prove that the set $\boldsymbol{F_1 = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}}$ (endowed with the addition and multiplication inherited from $\mathbb{R}$) is a field**

- **Closure under addition and multiplication**

First, we have to prove that F_1 is closed under the binary operations $+$ and $\times$, i.e. $\times: F_1 \times F_1 \to F_1$ and $+: F_1 \times F_1 \to F_1$.

Let $x_1, x_2 \in F_1$ with $x_1 = a + b\sqrt{3},\ x_2 = c + d\sqrt{3}$ and $a, b, c, d \in \mathbb{Q}$. Then:

$$\begin{aligned}
x_1 + x_2 &= \left(a + b\sqrt{3}\right) + \left(c + d\sqrt{3}\right) \\
&= \left(a + b\sqrt{3}\right) + \left(d\sqrt{3} + c\right) && \textit{by commutative property of addition on } \mathbb{R} \\
&= a + \left(b\sqrt{3} + d\sqrt{3}\right) + c && \textit{by associativity of addition on } \mathbb{R} \\
&= a + (b + d) \cdot \sqrt{3} + c && \textit{because addition and multiplication are distributive operations on } \mathbb{R} \\
&= (a + c) + (b + d)\sqrt{3} && \textit{by commutative and associative property of addition on } \mathbb{R}
\end{aligned}$$

Thus, $x_1 + x_2 = (a + c) + (b + d)\sqrt{3} \in F_1$ because $e = a + c \in \mathbb{Q}$ and $f = b + d \in \mathbb{Q}$ $\Longrightarrow F_1$ closed under addition.

$$\begin{aligned}
x_1 \cdot x_2 &= \left(a + b\sqrt{3}\right) \cdot \left(c + d\sqrt{3}\right) \\
&= a \cdot \left(d\sqrt{3} + c\right) + b\sqrt{3} \cdot (c + d\sqrt{3}) && \textit{because addition and multiplication are distributive operations on } \mathbb{R} \\
&= ad\sqrt{3} + ac + b\sqrt{3}c + b\sqrt{3}d\sqrt{3} && \textit{because addition and multiplication are distributive operations on } \mathbb{R} \\
&= \left(ad\sqrt{3} + bc\sqrt{3}\right) + ac + bd\sqrt{3}\sqrt{3} && \textit{by associativity and commutative property of addition and multiplication on } \mathbb{R} \\
&= (ad + bc)\sqrt{3} + ac + 3bd && \textit{because addition and multiplication are distributive operations on } \mathbb{R} \\
&= (ac + 3bd) + (ad + bc)\sqrt{3} && \textit{by commutative property and associativity of addition on } \mathbb{R}
\end{aligned}$$

Thus, $x_1 \cdot x_2 = (ac + 3bd) + (ad + bc)\sqrt{3} \in F_1$ because $e = ac + 3bd \in \mathbb{Q}$ and $f = ad + bc \in \mathbb{Q} \Longrightarrow F_1$ closed under multiplication.

- **$\boldsymbol{F1}$: Commutative property for addition and multiplication**

Let $x_1, x_2 \in F_1$ with $x_1 = a + b\sqrt{3},\ x_2 = c + d\sqrt{3}$ and $a, b, c, d \in \mathbb{Q}$. Then:

$$\begin{aligned}
x_1 + x_2 &= (a, b) + (c, d) \\
&= (a + c, b + d) && \textit{As proved above} \\
&= (c + a, d + b) && \textit{by commutative property of addition on } \mathbb{R} \\
&= (c, d) + (a, b) \\
&= x_2 + x_1
\end{aligned}$$

Since $x_1 + x_2 = x_2 + x_1$, addition is commutative on F_1.

$$
\begin{aligned}
x_1 \cdot x_2 &= (a,b)\cdot(c,d) \\
&= (ac+3bd, ad+bc) && \textit{As proved above} \\
&= (ca+3db, da+cb) && \textit{by commutative property of multiplication on } \mathbb{R} \\
&= (c,d)\cdot(a,b) \\
&= x_2 \cdot x_1
\end{aligned}
$$

Since $x_1 \cdot x_2 = x_2 \cdot x_1$, multiplication is commutative on F_1.

- ***F*2: Associativity for addition and multiplication**

Let $x_1, x_2, x_3 \in F_1$ with $x_1 = a + b\sqrt{3},\ x_2 = c + d\sqrt{3}, x_3 = e + f\sqrt{3}$ and $a,b,c,d,e,f \in \mathbb{Q}$. Then:

$$
\begin{aligned}
(x_1+x_2)+x_3 &= \big((a,b)+(c,d)\big)+(e,f) \\
&= (a+c, b+d)+(e,f) \\
&= \big((a+c)+e, (b+d)+f\big) \\
&= \big(a+(c+e), b+(d+f)\big) && \textit{by associativity of addition on } \mathbb{R} \\
&= (a,b)+(c+e, d+f) \\
&= (a,b)+\big((c,d)+(e,f)\big) \\
&= x_1+(x_2+x_3)
\end{aligned}
$$

Since $(x_1+x_2)+x_3 = x_1+(x_2+x_3)$, addition is associative on F_1.

$$
\begin{aligned}
(x_1\cdot x_2)\cdot x_3 &= \big((a,b)\cdot(c,d)\big)\cdot(e,f) \\
&= (ac+3bd, ad+bc)\cdot(e,f) \\
&= \big((ac+3bd)\cdot e + 3\cdot(ad+bc)\cdot f, (ac+3bd) \\
&\qquad \cdot f + (ad+bc)\cdot e\big) \\
&= (ace+3bde+3adf+3bcf, acf+3dbf && \textit{addition and} \\
&\qquad + ade + bce) && \textit{multiplication are} \\
& && \textit{distributive on } \mathbb{R} \\
&= (ace+3adf+3bde+3bcf, acf+ade && \textit{associativity of} \\
&\qquad + 3dbf + bce) && \textit{addition on } \mathbb{R} \\
&= (ace+a3df+3bde+3bcf, acf+ade && \textit{associativity of} \\
&\qquad + b3df + bce) && \textit{multiplication on } \mathbb{R} \\
&= \big(a\cdot(ce+3df)+3\cdot b\cdot(de+cf), a && \textit{addition and} \\
&\qquad \cdot(cf+de)+b\cdot(3df+ce)\big) && \textit{multiplication are} \\
& && \textit{distributive on } \mathbb{R} \\
&= (a,b)\cdot(ce+3df, cf+de) \\
&= (a,b)\cdot\big((c,d)\cdot(e,f)\big) \\
&= x_1\cdot(x_2\cdot x_3)
\end{aligned}
$$

Since $(x_1\cdot x_2)\cdot x_3 = x_1\cdot(x_2\cdot x_3)$, multiplication is associative on F_1.

- ***F*3: Existence of identity elements for addition and multiplication**

Let $x_1, 0_{F_1} \in F_1$ with $x_1 = a + b\sqrt{3},\ 0_{F_1} = 0 = 0 + 0\cdot\sqrt{3}$ and $a, b, 0 \in \mathbb{Q}$. Then:

$$
\begin{aligned}
x_1 + 0_{F_1} &= (a,b)+(0,0) \\
&= (a+0, b+0) \\
&= (a,b) \\
&= x_1 && \textit{by definition of 0 on } \mathbb{R}
\end{aligned}
$$

Therefore, $0_{F_1} = (0,0) \in F_1$ is the additive identity element for F_1

Let $x_1, 1_{F_1} \in F_1$ with $x_1 = a + b\sqrt{3},\ 0_{F_1} = 1 = 1 + 0 \cdot \sqrt{3}$ and $a, b, 1,0 \in \mathbb{Q}$. Then:

$$\begin{aligned} x_1 \cdot 1_{F_1} &= (a, b) + (1,0) \\ &= (a \cdot 1 + 3 \cdot b \cdot 0, a \cdot 0 + b \cdot 1) \\ &= (a, b) && \textit{by definition of 0 and 1 on } \mathbb{R} \\ &= x_1 \end{aligned}$$

Therefore, $1_{F_1} = (1,0) \in F_1$ is the multiplicative identity element for F_1. Moreover, we have $(0,0) \neq (1,0) \Leftrightarrow 0_{F_1} \neq 1_{F_1}$.

Finally, $\exists 1_{F1}, 0_{F_1} \in F_1$ such that $\forall a \in F_1, a + 0_{F_1} = a \wedge a \cdot 1_{F_1} = a$ with $0_{F_1} \neq 1_{F_1}$.

- **$F4$: Existence of inverses for addition and multiplication**

Let $x_1, x_2 \in F_1$ with $x_1 = a + b\sqrt{3},\ x_2 = c + d\sqrt{3}$ and $a, b, c, d \in \mathbb{Q}$. Then:

$$\begin{aligned} x_1 + x_2 = 0_{F_1} &\Leftrightarrow (a, b) + (c, d) = (0,0) \\ &\Leftrightarrow (a + c, b + d) = (0,0) \\ &\Leftrightarrow a + c = 0 \wedge b + d = 0 \\ &\Leftrightarrow c = -a \wedge d = -b \\ &\Leftrightarrow x_2 = (-a, -b) \end{aligned}$$

Moreover, $a, b \in \mathbb{Q} \Longrightarrow -a, -b \in \mathbb{Q} \Longrightarrow x_2 \in F_1$.
Finally, $\forall x_1 \in F_1, \exists x_2 = (-a, -b) \in F_2$ such that $x_1 + x_2 = 0_{F1}$.

Let $x_1, x_2 \in F_1$ with $x_1 = a + b\sqrt{3},\ x_2 = c + d\sqrt{3}$ and $a, b, c, d \in \mathbb{Q}$ such that $x_1 \neq 0_{F_1} \wedge x_2 \neq 0_{F_1}$. Then:

$$\begin{aligned} x_1 \cdot x_2 = 1_{F_1} &\Leftrightarrow (a, b) + (c, d) = (1,0) \\ &\Leftrightarrow (ac + 3bd, ad + bc) = (1,0) \\ &\Leftrightarrow \begin{cases} ac + 3bd = 1 \\ ad + bc = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} c = \dfrac{1 - 3bd}{a} \\ ad + b \cdot \left(\dfrac{1 - 3bd}{a}\right) = 0 \end{cases} && \textit{Assuming that } a \neq 0 \textit{ WLOG (since } x_1 \neq 0 \Longrightarrow a \neq 0 \vee b \neq 0) \\ &\Leftrightarrow \begin{cases} c = \dfrac{1 - 3bd}{a} \\ \dfrac{a^2 d}{a} + \dfrac{b - 3b^2 d}{a} = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} c = \dfrac{1 - 3bd}{a} \\ a^2 d + b - 3b^2 d = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} c = \dfrac{1 - 3bd}{a} \\ a^2 d - 3b^2 d = -b \end{cases} \\ &\Leftrightarrow \begin{cases} c = \dfrac{1 - 3bd}{a} \\ d \cdot (a^2 - 3b^2) = -b \end{cases} \end{aligned}$$

Moreover: $a^2 - 3b^2 = 0 \Longrightarrow a^2 = 3b^2 \Longrightarrow a = \sqrt{3} \cdot b \vee a = -b \cdot \sqrt{3}$. But, $\nexists a, b \in \mathbb{Q}$ such that $a = \sqrt{3} \cdot b \vee a = -b \cdot \sqrt{3}$ if $a \neq 0 \vee b \neq 0$. Therefore, $\forall a, b \in \mathbb{Q}$ such that $a \neq 0 \vee b \neq 0$, $a^2 - 3b^2 \neq 0$.

$$\Leftrightarrow \begin{cases} c = \dfrac{1 - 3bd}{a} \\ d = -\dfrac{b}{(a^2 - 3b^2)} \end{cases}$$

$$\Leftrightarrow \begin{cases} c = \dfrac{1 - 3b \cdot \left(\dfrac{-b}{a^2 - 3b^2}\right)}{a} \\ d = -\dfrac{b}{(a^2 - 3b^2)} \end{cases}$$

$$\Leftrightarrow \begin{cases} c = \dfrac{\dfrac{1}{a^2 - 3b^2} \cdot \left(a^2 - 3b^2 - 3b \cdot (-b)\right)}{a} \\ d = -\dfrac{b}{(a^2 - 3b^2)} \end{cases}$$

$$\Leftrightarrow \begin{cases} c = \dfrac{a^2 - 3b^2 + 3b^2}{(a^2 - 3b^2) \cdot a} \\ d = -\dfrac{b}{(a^2 - 3b^2)} \end{cases}$$

$$\Leftrightarrow \begin{cases} c = \dfrac{a^2}{(a^2 - 3b^2) \cdot a} \\ d = -\dfrac{b}{(a^2 - 3b^2)} \end{cases}$$

$$\Leftrightarrow x_2 = \left(\frac{a}{(a^2 - 3b^2)}, -\frac{b}{(a^2 - 3b^2)}\right)$$

Moreover, $a, b \in \mathbb{Q} \Longrightarrow \frac{a}{(a^2 - 3b^2)} \in \mathbb{Q} \wedge -\frac{b}{(a^2 - 3b^2)} \in \mathbb{Q} \Longrightarrow x_2 \in F_1$.
Finally, $\forall x_1 \in F_1, \exists x_2 \in F_2$ such that $x_1 \cdot x_2 = 1_{F1}$.

- **$F5$: Distributive property of addition and multiplication**

Let $x_1, x_2, x_3 \in F_1$ with $x_1 = a + b\sqrt{3}$, $x_2 = c + d\sqrt{3}$, $x_3 = e + f\sqrt{3}$ and $a, b, c, d, e, f \in \mathbb{Q}$. Then:

$$\begin{aligned} x_1 \cdot (x_2 + x_3) &= (a, b) \cdot \left((c, d) + (e, f)\right) \\ &= (a, b) \cdot (c + e, d + f) \\ &= \left(a \cdot (c + e) + 3 \cdot b \cdot (d + f), a \cdot (d + f) + b \cdot (c + e)\right) \\ &= (ac + ae + 3bd + 3bf, ad + af + bc + be) \\ &= \left((ac + 3bd) + (ae + 3bf), (ad + bc) + (af + be)\right) \\ &= (ac + 3bd, ad + bc) + (ae + 3bf, af + be) \\ &= (a, b) \cdot (c, d) + (a, b) \cdot (e, f) \\ &= x_1 \cdot x_2 + x_1 \cdot x_3 \end{aligned}$$

addition and multiplication are distributive on $\mathbb{R}$
commutative property and associativity of addition on $\mathbb{R}$

Finally, $\forall x_1, x_2, x_3 \in \mathbb{Q},\ x_1 \cdot (x_2 + x_3) = x_1 \cdot x_2 + x_1 \cdot x_3$

To conclude, F_1 is a set endowed with two binary operations $\times: F_1 \times F_1 \to F_1$ and $+: F_1 \times F_1 \to F_1$, which has two special elements $0_{F_1}, 1_{F_1} \in F_1$ such that $0_{F_1} \neq 1_{F_1}$ and satisfies the five properties of a field. Therefore, F_1 is a field.

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5.2. Is the set $F_2 = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\}$ (with the same addition and multiplication) also a field ?

Suppose that F_2 is a field.

Notice that $1_{F_2} = (1,0) \in F_2$ with $1,0 \in \mathbb{Z}$ is the identity element for multiplication since $\forall x = (a,b) \in F_2$ with $a, b \in \mathbb{Z}$, we have:

$$\left(a + b\sqrt{3}\right) \cdot \left(1 + 0 \cdot \sqrt{3}\right) = \left(a + b\sqrt{3}\right) \cdot 1 = \left(a + b\sqrt{3}\right) \Leftrightarrow x \cdot 1_{F_1} = x \ \ \forall x \in F_2$$

Moreover, we also note that $0_{F_2} = (0,0) \in F_2$ with $0 \in \mathbb{Z}$ is the identity element for addition since $\forall x = (a,b) \in F_2$ with $a, b \in \mathbb{Z}$, we have:

$$\left(a + b\sqrt{3}\right) + \left(0 + 0 \cdot \sqrt{3}\right) = \left(a + b\sqrt{3}\right) + 0 = \left(a + b\sqrt{3}\right) \Leftrightarrow x + 0_{F_2} = x \ \forall x \in F_2$$

Since $1 \in \mathbb{Z}$, $x_1 = (1,1) \in F_1$. Moreover, since $x_1 \in \mathbb{Z}, \exists x_2 \in F_2$ such that $x_2 = (a,b)$ with $a, b \in \mathbb{Z}$ and $x_1 \cdot x_2 = 1_{F_2}$ such that $x_2 \neq 0_{F_2}$. Therefore, we have:

$$\begin{aligned}
\left(1 + \sqrt{3}\right) \cdot \left(a + b\sqrt{3}\right) = 1 \quad &\Leftrightarrow a + a\sqrt{3} + b\sqrt{3} + 3b = 1 \\
&\Leftrightarrow a + 3b + (a + b) \cdot \sqrt{3} = 1 \\
&\Leftrightarrow \begin{cases} a + 3b = 1 \\ a + b = 0 \end{cases} \ since\ a, b \in \mathbb{Z} \\
&\Leftrightarrow \begin{cases} -b + 3b = 1 \\ a = -b \end{cases} \ because\ a \neq 0 \lor b \neq 0 \\
&\Leftrightarrow \begin{cases} b = \dfrac{1}{2} \\ a = -b \end{cases} \\
&\Leftrightarrow \begin{cases} b = \dfrac{1}{2} \\ a = -\dfrac{1}{2} \end{cases}
\end{aligned}$$

Therefore, $x_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$ is the inverse of $(1,1)$. This contradicts the fact that $a, b \in \mathbb{Z}$ for $x_2 = (a,b) \in F_2$. Thus, for $x_1 = (1,1) \in F_2, \nexists x_2 \in F_2$ such that $x_1 \cdot x_2 = 1_{F_2}$. Hence F_2 is not a field.

■