Oct 22, 2012

Read Along: 3.1, 3.3~3.5, 7

<u>Constant Coefficient Linear Homogeneous ODES</u>

$$Ly = \sum_{k=0}^{n} a_k y^{(k)} = 0 \quad a_k \in \mathbb{R} \qquad P(\alpha) = \sum_{k=0}^{n} a_k \alpha^k, \ D = \frac{d}{dx}$$

expect an n-dim v.s of sol's

$Ly = P(D)y = 0$

Guess $y = e^{\alpha x}$, in this case

$(*) \cdots P(D) e^{\alpha x} = P(\alpha) \underline{e^{\alpha x}} = 0 \Rightarrow P(\alpha) = 0$

$\alpha = \beta \pm i\gamma \Rightarrow e^{(\beta \pm i\gamma)x} = e^{\beta x}(\cos\gamma x \pm i \sin\gamma x)$

If p has a multiple root.

<u>example</u>

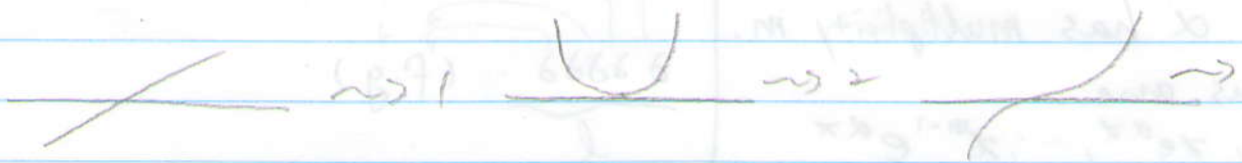
$y'' - 2y' + y = 0$

$p(\alpha) = \alpha^2 - 2\alpha + 1 = (\alpha - 1)^2 = 0$

$\alpha_{1,2} = 1$

first solution : e^x

second solution : xe^x



α is of multiplicity $m \Leftrightarrow P(\alpha) = 0, P'(\alpha) = 0, \cdots, P^{(m-1)}(\alpha) = 0, P^{m}(\alpha) \neq 0$

Take (*) and apply

$\frac{\partial}{\partial \alpha} = \partial_\alpha \qquad D \sim \frac{\partial}{\partial x}$

$\frac{\partial}{\partial \alpha} \cdot \frac{\partial}{\partial x} = \frac{\partial}{\partial x}\frac{\partial}{\partial \alpha} \qquad \partial_\alpha D = D \partial_\alpha$

LHS $(*)$: apply ∂_α

$$\partial_\alpha P(D) e^{\alpha x} = P(D) \partial_\alpha e^{\alpha x} = P(D) x e^{\alpha x}$$

RHS $(*)$: apply ∂_α

$$\partial_\alpha (P(\alpha) e^{\alpha x}) = P'(\alpha) e^{\alpha x} + P(\alpha) x e^{\alpha x}$$

$$P(D) x e^{\alpha x} = (P' + xP) e^{\alpha x} \overset{\text{if } \alpha \text{ is multiple root}}{=} 0$$

In general,

$$\partial_\alpha^l (LHS) = \partial_\alpha^l P(D) e^{\alpha x} = P(D) \partial_\alpha^l e^{\alpha x} = P(D) x^l e^{\alpha x}$$

$$\partial_\alpha^l (RHS) = \partial_\alpha^l (P \cdot e^{\alpha x})$$

$$= \sum_{j=0}^{l} \binom{l}{j} P^{(j)} x^{l-j} e^{\alpha x}$$

$$\underset{l<m}{=} 0$$

multiplicity of α

Aside

$$\partial^l (f \cdot g)$$

$$= \sum_{j=0}^{l} \binom{l}{j} (\partial^j f)(\partial^{l-j} g)$$

proof

By induction

$$\underbrace{\partial\partial\partial\partial\partial \cdots}_{l} (f \cdot g)$$

$$\partial(fg) = (\partial f) g + f(\partial g)$$

$$\partial^2(fg) = \partial((\partial f) g + f(\partial g))$$

$$= \partial^2 f g + \partial f \partial g + \partial f \partial g + f \partial^2 g$$

$$= (\partial^2 f) g + 2(\partial f)(\partial g) + f(\partial^2 g)$$

So if α has multiplicity m, solutions are

$$\underbrace{e^{\alpha x}, x e^{\alpha x}, \ldots, x^{m-1} e^{\alpha x}}_{m \text{ solutions}}$$

Examples

1. $y''' - 3y'' + 3y' - y = 0$

$$P = \alpha^3 - 3\alpha^2 + 3\alpha - 1$$
$$= (\alpha - 1)^3$$
$$= 0$$

Solutions are $e^x, xe^x, x^2 e^x$

2. $y^{(4)} = 0$

$P = \alpha^4 = 0$ is a root of multiplicity 4

Solutions are $e^{0x}, xe^{0x}, x^2 e^{0x}, x^3 e^{0x}$

$\Rightarrow \quad 1, x, x^2, x^3$

All solutions : polynomial of degree 3 in x

Reduction of Order

Suppose y_1 solves $y'' + py' + qy = 0$

p, q are functions of x

Substitute $y = y_1 \cdot v$

$\cancel{y_1'' v} + 2y_1' v' + y_1 v'' + \cancel{py_1' v} + py_1 v' + \cancel{q y_1 v} = 0$

$y_1 v'' + (2y_1' + py_1) v' = 0$

$y_1 u' + (2y_1' + py_1) u = 0 \qquad u = v'$

$\Rightarrow \quad u = 1$

$\Rightarrow \quad v = x$

Exercises

1. Verify previous example using new techique
2. How is this related to the algebraic equation of order

Non-Homogeneous Case Using "undetermined coefficients"
"guessing intelligently"

example

$y'' - 3y' - 4y = 2\sin x$

$\alpha^2 - 3\alpha - 4 = 0 \Rightarrow \alpha_{1,2} = 4, -1$

Guess $A\sin x + B\cos x$

$y'' = -A\sin x - B\cos x$

$-3y' = 3B\sin x - 3A\cos x$

$-4y = -4A\sin x - 4B\cos x$

$2\sin x = (-5A + 3B)\sin x + (-3A - 5B)\cos x$

$\left.\begin{array}{l} -5A + 3B = 2 \\ -3A - 5B = 0 \end{array}\right) \Rightarrow A = -\frac{5}{17}, \; B = \frac{3}{17}$