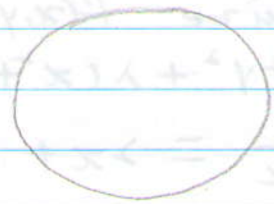
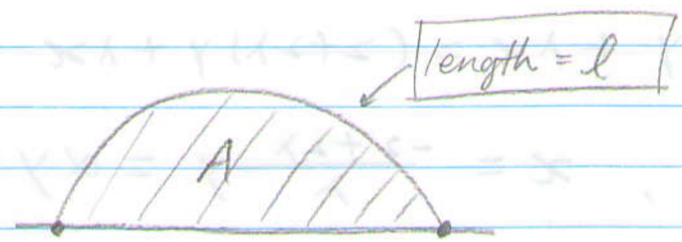
Oct 9, 2012 Mat 267

Isoperimetric Inequality

perimeter $= \ell$

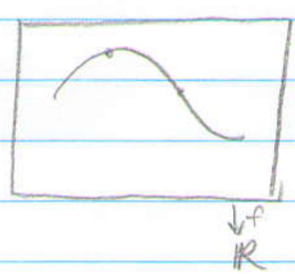
shape with maximum area is 



minimize A

Warmup Problem

$F: \mathbb{R}^2 \longrightarrow \mathbb{R}$



constant

Maximize f on the curve defined by $g(x,y) = 0$

1. Messy way
   Solve for y as a function of x. sub. into $f(x, y(x))$ and maximize

"Lagrange multiplier"

2. Use "Lagrange Multipliers"

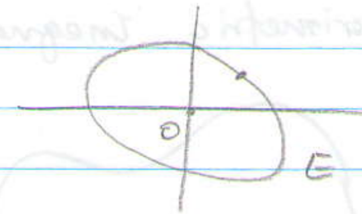
$$h_\lambda(x,y) = f(x,y) + \lambda g(x,y)$$

Solve : $\nabla h_\lambda = 0$: $\frac{\partial h_\lambda}{\partial x} = 0$, $\frac{\partial h_\lambda}{\partial y} = 0$ & $g(x,y) = 0$

example

$g(x,y) = 0$

$g(x,y) = x^2 + xy + y^2 - 1$



Find point closest to O on this ellipse, i.e. maximize $f(x,y) = \sqrt{x^2+y^2}$ on E

$\Rightarrow$ Maximize $f(x,y) = x^2+y^2$ on E

$h_\lambda = x^2+y^2+\lambda(x^2+xy+y^2-1)$

$$0 = \frac{\partial h}{\partial x} = 2x + 2\lambda x + \lambda y = (2+2\lambda)x + \lambda y$$

$$0 = \frac{\partial h}{\partial y} = 2y + 2\lambda y + \lambda x = (2+2\lambda)y + \lambda x$$

$$\Rightarrow y = \underbrace{-\frac{2+2\lambda}{\lambda}}_{\alpha} x, \quad x = \frac{-2+2\lambda}{\lambda} y = \alpha y$$

$\Rightarrow y = \alpha x = \alpha^2 y$

$\Rightarrow$ solution iff $\alpha = \pm 1$

$\Rightarrow y = \pm x$

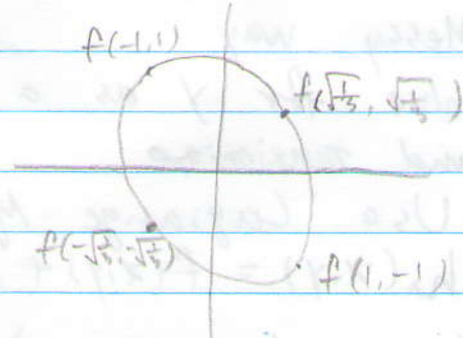
case $x = y$

$g(x,y) = 0$

$= g(x,x)$

$= 3x^2 - 1$

$\Rightarrow x = \pm\sqrt{\frac{1}{3}}, \quad y = \pm\sqrt{\frac{1}{3}}$

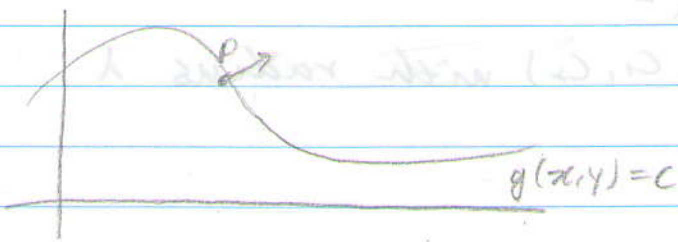


case $y = -x$

$g(x,y) = x^2 - 1 = 0$

$\Rightarrow x = \pm 1, \quad y = \mp 1$

Why?



$\xrightarrow{f} \mathbb{R}$

$\nabla f \perp$ curve

$\nabla g \perp$ curve

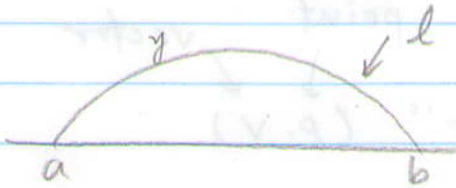
$\Rightarrow \nabla f = \lambda \nabla g$

Two properties of ∇f

1. Points in the direction in which f rises most quickly
2. ∇g is perpendicular to equal height curves of g

example

Maximize A subject to arc-length $= l$.



$$J(y) = \int_a^b \underbrace{y}_{F}\, dx \quad \text{subject to}$$

$$L(y) = \int_a^b \underbrace{\sqrt{1+y'^2}}_{G}\, dx$$

$$H_\lambda = F + \lambda G$$
$$= y + \lambda\sqrt{1+y'^2}$$

Solve E-L for H_λ

$$\boxed{F_y - \frac{d}{dx} F_{y'} = 0}$$

E-L (H_λ),

$$1 - \frac{d}{dx}\left(\lambda \frac{y'}{\sqrt{1+y'^2}}\right) = 0 \Rightarrow \lambda \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 1$$

$$\lambda \frac{y'}{\sqrt{1+y'^2}} = x - C_1$$

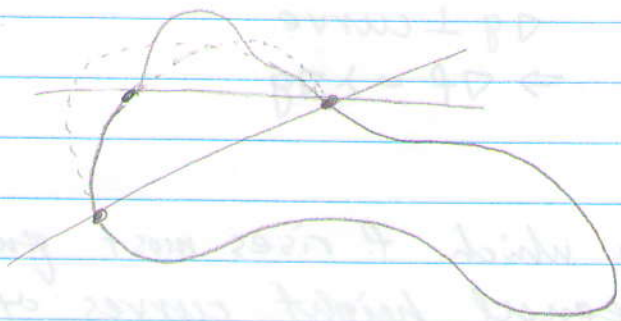
Solve for y' & get

$$y' = \frac{x - C_1}{\sqrt{\lambda^2 - (x-C_1)^2}}$$

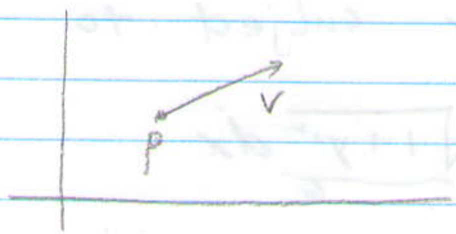
$$y - c_2 = \sqrt{\lambda^2 - (x-c_1)^2}$$

$$(x-c_1)^2 + (y-c_2)^2 = \lambda^2$$

A circle centred at (c_1, c_2) with radius λ



$f : \mathbb{R}^n \longrightarrow \mathbb{R}$



"tangent vector" (p, v)

point → p, vector → v

"The directional derivative of f at p in the direction of v"

Precisely,

$$
\begin{aligned}
D_v f &= D_{(p,v)} f \\
&= \frac{d}{d\varepsilon} f(p + \varepsilon v)\Big|_{\varepsilon=0} \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot v_i \\
&= (\nabla f) \cdot v
\end{aligned}
$$

$$\boxed{\begin{array}{c} \mathbb{R}_\varepsilon \longrightarrow \mathbb{R}^n \xrightarrow{f} \mathbb{R} \\ \varepsilon \longmapsto p + \varepsilon v \end{array}}$$