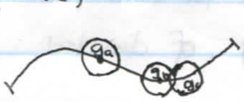
The partition of utility lemma: (Big theorm).
Given a collection $\mathcal{A}$ of open sets in $\mathbb{R}^n$ whose overall union is $A = \bigcup_{U \in \mathcal{A}} U$, there exist a sequence $\{\phi_i\}$ of non-negative compactly supported C^∞ functions s.t. $\text{supp}(\phi) = \{x : \phi(x) \neq 0\}$.

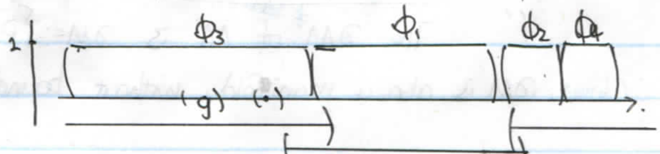
Compactly supported: $\text{supp}(\phi)$ is compact.
Partitions of Unity Lemma: Given a collection $\mathcal{A}$ of open sets in $\mathbb{R}^k$ whose overall union is $A = \bigcup_{U \in \mathcal{A}} U$, there exists a seq. $\{\phi_i\}$ of non-neg compactly-supp. C^∞ fcns s.t.: 1. $\forall i. \exists U \in \mathcal{A}$ s.t. $\text{supp}(\phi_i) \subset U$.
2. Each $\text{supp}\,\phi_i$ is contained in one $U \in A$. 3. $\sum \phi_i(x) = 1$.

Jan 23.

$\partial M = \{\text{bndry pts}\} = \{p \in M : \text{for some patch } \alpha,\ p = \alpha(q) \text{ with } q \in \partial H^k := \mathbb{R}^{k-1} \times \{0\}.\}$.
Thm. With M as before, ∂M is a $(k-1)$-manifold of class C^r with no boundary.
Def. $f : S \subset \mathbb{R}^k \to \mathbb{R}$ is C^r differentiable if $\forall p \in S$, $\exists$ nbd U of p is $\mathbb{R}^k$ $\exists g : U \to \mathbb{R}$ s.t.
1. g is C^r. 2. $g(x) = f(x)$ for $x \in U \cap S$.

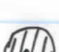
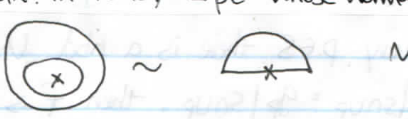
Thm.(2). Let S be an arbitrary subset of $\mathbb{R}^k$. A function $f : S \to \mathbb{R}^n$ is said to be of class C^r iff there exists an open sub set U of $\mathbb{R}^k$ containing S and a C^r fcn $g : U \to \mathbb{R}^n$ such that $g(x) = f(x)$ for every $x \in S$.
proof: (of thm 2). $(\Leftarrow)$ ✓
$(\Rightarrow)$ hard.





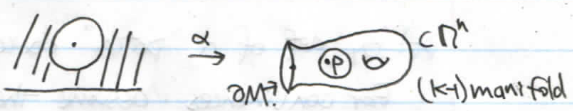
Partition of units: "division of labor".
Let $\mathcal{A} = \{U_{\text{open}} : F|_{S \cap U}$ has a diff able extension to U $\exists g : U \to \mathbb{R}^n$ 1. g is C^r. 2. $g(x) = f(x)$ for $x \in S \cap U\}$.
Then $A = \bigcup_{U \in \mathcal{A}} U$ is an open set containing S.
Find a POI. $\{\phi_i\}$ subordinate to $\mathcal{A}$. For each i find $U_i \in \mathcal{A}$ & a C^r. $g_i : U \to \mathbb{R}^n$ s.t.
$g_i|_{U_i \cap S} = f|_{U_i \cap S}$ ~~wrong~~ $g(x) = \sum_{i=1}^{\infty} \phi_i(x) \cdot g(x)$ extended by 0 to A.
g is diff able $g(x) = f(x)$ for $x \in S$. □

Jan 25

Q. Is  homeo to ? (No). Precisely, is $B^k = \{x \in \mathbb{R}^k : \|x\| < 1\}$ homeo to $HB^k = \{x \in B^k : x_k \geq 0\}$?
$k=1$. Is $(-1,1) \sim [0,1)$. $(\quad) \sim [\quad)$?
Soln: in RHS, $\exists$ pt whose nomal keeps connidivits, but not on left.
$k=2$.  MAT 327.

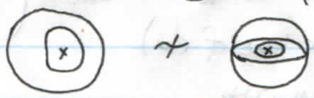
$k=3$. hard.

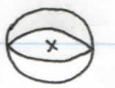
Aside. Is  homeo to  ? (No).



Is B^k homeo to B^n for $n \neq k$?

$k=1$. $n>1$. $(\longleftarrow\!\!\bullet\!\!\longrightarrow) \not\approx$  pf. remove a pt. interval becomes disconnected !

$k=2$. $n>2$.  ? "Simple connectivity" : near the end of MAT 327 when π_1.

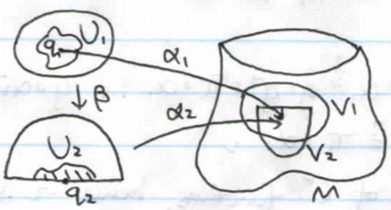
$k=3$. $n>k$.  MAT 1301 near the end. hard.

Now homeo $\to$ diffsomorphic : X diffso to Y. means $\exists \alpha: X \to Y$. diffable. with diff. inverse. (much easier).

$B^k \not\approx HB^k$ $\quad B^k \not\approx B^n \quad k \neq n$.

If $\alpha: B^k \to B^n$ $(k \neq n)$ diffable with α^{-1} also diffable. then $(d\alpha)^{-1} = d(\alpha^{-1})$

In particular, $(d\alpha)^{-1}$ exists, but no $n \times k$ matrix is invertible.



$V_1 \cap V_2$ is an open nbd $\neq$ of p in M.

Let $(i=1,2)$ $U_i' = \alpha_i^{-1}(V_1 \cap V_2)$.

The U_i' are open nbds of q_j in $\mathbb{R}^k / H^k$

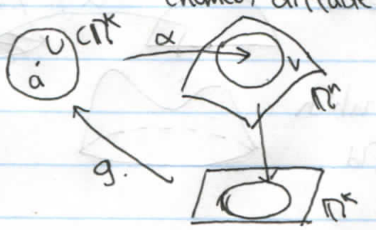
Let $\beta: \alpha_2^{-1}\alpha_1 : U_1' \to U_2'$ is 1-1 & onto, is homeo.

By 1301 get a contradiction.

If we know that α_2^{-1} is diffable, we'd done without 1301.

Is α_2^{-1} diffable ?

Proposition: If M^k is a class C^r m-fld in $\mathbb{R}^n$ and $\alpha: U \subset \mathbb{R}^k \to V \subset M^k$ is a coord. match. (homeo, diffable. $d\alpha$ has max rank). then $\alpha^{-1}: V^{\subset \mathbb{R}^n} \to U$ is also C^r.



$d\alpha(q)$ is of rank k.

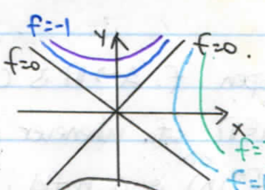
For convenince. assume the first k runs of $d\alpha(q)$ are indep.

Jan 27

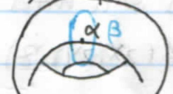
$\mathbb{R}^2_{x,y} \to \mathbb{R}^3 \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ x^2-y^2 \end{pmatrix}$

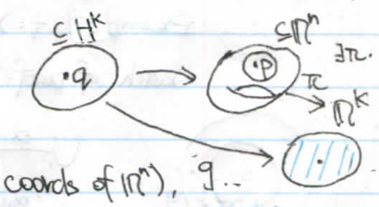
let $f(x,y) = x^2-y^2$; $x^2-y^2=0 \iff (x+y)(x-y)=0$.

i.e. $x=y$ or $x=-y$.



$\mathbb{R}^2_{\alpha,\beta} \longrightarrow \mathbb{R}^3 \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto \begin{pmatrix} \cos\alpha\,(2+\cos\beta) \\ \sin\alpha\,(2+\cos\beta) \\ \sin\beta \end{pmatrix}$

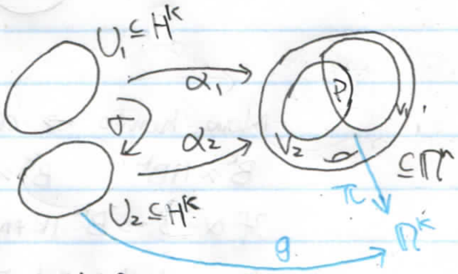




Thm: Given a patch $\alpha: U^{\subset H^k} \to M^{\subset \mathbb{R}^n}$ and any $p = \alpha(a)$. there exists a projection $\pi: \mathbb{R}^n \to H^k$ (defined by forgetting $(n-k)$ of the coords of $\mathbb{R}^n$), g.. such that $g = \pi \circ \alpha$. then g is a diffso in the vinity of q.

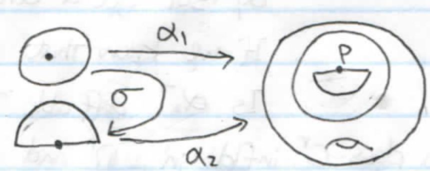
Pf: By def of a "patch" $d\alpha(q)$ has rank k, $n\{\overbrace{\left(d\alpha(q) \right)}^{k}$ so it has k indep. rows.
For convinences, assume these are the first k rows, and let $\pi: \mathbb{R}^n \to \mathbb{R}^k$ be given by dropping the last $(n-k)$ coords of a vector in $\mathbb{R}^n$. $d\pi = \overbrace{\left(I_k \mid 0 \right)}^{n}\}k$
Set $g = \pi\alpha$, $dg = (\underset{k}{I} \mid \underset{n-k}{0})(d\alpha) = $ (first k rows of $d\alpha$)
So $dg(q)$ has k indep rows. so $dg(q)$ is invertible.
By the IFT, g is a diffso near q.

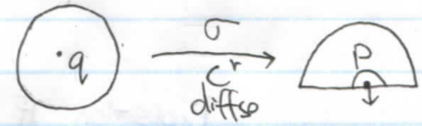
Corollary: "Trasition function are C^r"



$\sigma = \alpha_2^{-1} \circ \alpha_1$ on $\alpha_1^{-1}(V_1 \cap V_2)$.
Claim: σ is C^r
pf: find a projection: $\pi: \mathbb{R}^n \to \mathbb{R}^k$ s.t.
$g = \pi \circ \alpha_2$ is a diffso near some arbry pt. $p \in V_1 \cap V_2$.
Claim: $\sigma = \alpha_2^{-1} \circ \alpha_1 = g^{-1} \circ \pi \circ \alpha_1$ (near p)
pf: (by composing with g. nts $g \circ \alpha_2^{-1} \circ \alpha_1 \overset{?}{=} g \circ g^{-1} \circ \pi \circ \alpha_1$; $g \circ \alpha_2^{-1} \circ \alpha_1 \overset{?}{=} \pi \circ \alpha_1$
Recall $g = \pi \circ \alpha_2$, $\pi \circ \alpha_2 \circ \alpha_1^{-1} \circ \alpha_1 \overset{?}{=} \pi \circ \alpha_1$ ✓
So $\sigma = g^{-1} \circ \pi \circ \alpha_1$, is a composition of C^r functions, hence it is C^r as required.

Cor. ∂M make sense.



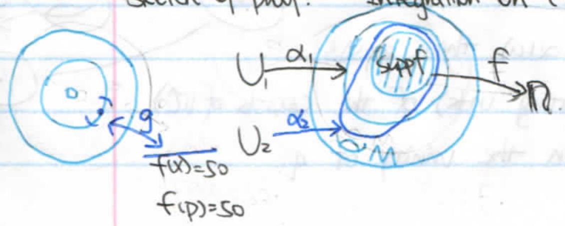


Easy. ∂M is itself a mfd, with no bdy. Sketch:

If $f: \mathbb{R}^n \to \mathbb{R}$ is C^r and t is a height at which df has no singularties, then $f^{-1}(t)$ is a mfld.

Jan 30.

Thm. If $A \subseteq \mathbb{R}^n$ is open. $f: A \to \mathbb{R}$ is C^r, then for most h, $f^{-1}(h)$ is a manifold. ($(n-1)$-dim. manifold)
Precisely: If $h \in \mathbb{R}$ s.t. whenever $p \in f^{-1}(h)$, $df(p)$ is of rank 1 ($\Leftrightarrow df(p) \neq 0$)
then $\underset{}{\overset{=N}{f^{-1}(h)}}$ is a mfld, so is $M = f^{-1}((-\infty, h])$. Finally, $\partial M = N$.
Example: $f = x^2 + y^2 + z^2$ $df \neq (2x, 2y, 2z)$. So spheres are mflds.
Sketch of proof: Integration on (compact) mflds.



In that case, define $\int_M f\,dV = \int_U (f \circ \alpha) V(d\alpha)$
Q: Does $\int_M f\,dV$, as defined above, depend on α?
$\int_{U_1} (f \circ \alpha_1) V(d\alpha_1) \overset{?}{=} \int_{U_2} (f \circ \alpha_2) V(d\alpha_2)$