proof: By the previous thm, $f(X)$ is compact, hence $f(X)$ is bdd so f is bdd and $f(X)$ is closed, so it contains its limit pts. so $\sup f(x) \in f(X) \Rightarrow \sup_x f(x) = f(x_0)$ for some x_0. $\square$

sets both closed & open.

Def. A space X is called "$\boxed{\text{connected}}$" if there are no clopen sets in X except $\emptyset$ & X.

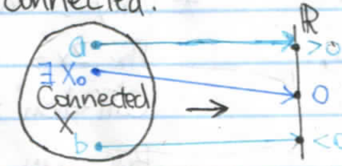
e.g. $X = (0,1) \cup (2,3) \subset \mathbb{R}$ is not connected because $(0,1)$ & $(2,3)$ is clopen in X.

Thm1 $A \subset \mathbb{R}$ is connected iff it is a generalised interival (open or closed, finite or not)

Thm2 X, Y connected $\Rightarrow X \times Y$ is connected.

Thm3 If X is connected & $f: X \to Y$ is cont. then $f(X)$ is connected.

Cor. The intermediate value thm.



Oct 3.

e.g. $\mathbb{R} \setminus \{257\}$ =

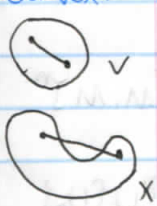
is not connected.

$X \cap (-\infty, 257) = (-\infty, 257] \cap X$ is clopen.

$X = A \cup B \quad A \cap B = \emptyset$. A & B are both open or both closed.

iff at least one of A & B is empty.

Convex.



$\forall a, b \in X \quad [a,b] \subset X$ i.e. X is convex.

proof (of thm 1). ($\Rightarrow$) Assume X is connected yet not convex (a general interval).

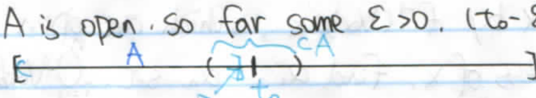
Namely $\exists a, b, c$ s.t. $a < b < c$, $a, b \in X$ yet $b \notin X$

then $X = \underbrace{(X \cap (-\infty, b))}_{A} \cup \underbrace{(X \cap (b, \infty))}_{B}$

A & B are open, $a \in A$, $b \in B$. $\Rightarrow X$ is not connected.

($\Leftarrow$) I'll only prove that $[0,1]$ is connected. Assume $[0,1] = A \cup A^c$ where A is clopen & non-trivial WLOG $0 \in A$.

Let $t_0 = \sup\{t \in [0,1] : [0,t] \in A\}$, then $t_0 \in A$ as A is closed but A is open. so for some $\varepsilon > 0$. $(t_0 - \varepsilon, t_0 + \varepsilon) \cap [0,1] \subset A$.



By def. of R find $r \in R$ s.t. $r \in (t_0 - \varepsilon, t_0)$

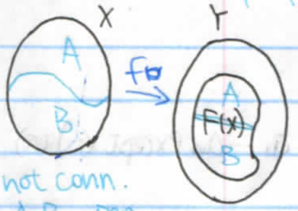
But then $\underbrace{[0, r)}_{A} \cup \underbrace{((t_0 - \varepsilon, t_0 + \varepsilon) \cap X)}_{A} \subset A \Rightarrow [0, t_0] \subset A$ so $t_0 \in A$

if $t_0 < 1$ then you can find some $s \in (t_0, t_0 + \varepsilon) \cap [0,1]$

then $[0, s] \subset A$ so $s \in R$. yet $s > t_0 = \sup R$ so $t_0 = 1$.

So $[0, t_0] = [0,1] \subset A$. So $A^c = \emptyset$ $\square$.

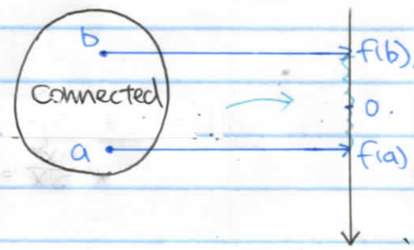
proof (of thm 2).



not conn.
A.B open.
$f^{-1}(A). f^{-1}(B)$ open

Assume $A \subset f(X)$ is clopen & not equal to $\emptyset$ or to $f(X)$.
then $f^{-1}(A)$ is clopen in X, and diff. from $\emptyset$ and X. contradicted.

proof (of thm 3).

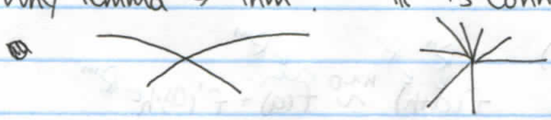


~~Cons~~ Consider $f(X)$. it is connected as an image of a connected set, $0 > f(a) \subset F(X)$. $0 < f(b) \in F(X)$.
So $0 \in f(X)$; by thm 1. $\Rightarrow \exists X_0 \in X$ s.t $f(X_0) = 0$.

Thm: If $f: \mathbb{R}^n \to \mathbb{R}$ conti. $\exists a, b \in \mathbb{R}^n$ s.t. $f(a) < 0 < f(b) \Rightarrow \exists X_0 \in \mathbb{R}$ s.t. $f(X_0) = 0$.

Lemma: If $X = \bigcup A_\alpha$ where 1. A_α is connected 2. $\bigcap A_\alpha \neq \emptyset$, then X is connected.

Why lemma $\Rightarrow$ thm? $\mathbb{R}^n$ is connected.



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Remainder: If $f: \mathbb{R} \to \mathbb{R}$ $a \in \mathbb{R}$ $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ (if the limit exists)
(really meant $f: A \to \mathbb{R}$, $a \in A \subset \mathbb{R}$. A contains a nbd of a.

Q: How should we define $f'(a)$ for $f: \mathbb{R}^n \to \mathbb{R}^m$

$$f_0(X, Y) = f_0\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} e^x \cos y \\ e^x \sin y \\ (1 + x^2 + y^2)^{-1} \end{pmatrix} \quad * f: \mathbb{R}^2 \to \mathbb{R}^3$$

*If $f: \mathbb{R} \to \mathbb{R}^m$ e.g. $r: \mathbb{R}_t \to \mathbb{R}^2_{x,y}$ $r(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ All is well.

Check: $r'(t) = \begin{pmatrix} (\cos t)' \\ (\sin t)' \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

$f: \mathbb{R}^n \to \mathbb{R}^m$ Naive def is a total failure. $f'(a) = \lim \frac{f(a+h) - f(a)}{h}$.
$\mathbb{R}^n$ $\mathbb{R}^n$

Attemp 1 "Directional Derivatives"

Def $f'(a; u) = \lim_{h \to 0} (f(a + hu) - f(a)) / h$. "dairectional derivative of f at a in the dirction of u"
(if the limit exists, otherwise derictional derivative not exist).

e.g. 0. $f'(a; 0) = 0$.

1. $f_0'(0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \cdots \in \mathbb{R}^3$

$$= \lim_{h \to 0} \left(f_0\begin{pmatrix} 0 \\ 0 \end{pmatrix} + h\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) - f_0\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Big) / h = \lim_{h \to 0} h^{-1} (f_0(h, h) - f_0(0, 0))$$

$$= \lim_{h \to 0} h^{-1} \left[\begin{pmatrix} e^h \cosh \\ e^h \sinh \\ (1 + 2h^2)^{-1} \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 2 \\ \ldots \end{pmatrix}$$

note: $\lim \frac{e^h \cosh - 1}{h} = (e^h \cosh)'_{h=0} =$

e.g. 2. $f:\mathbb{R}^n\to\mathbb{R}^m$ $\quad a\in\mathbb{R}^n$ $\quad u=e_i=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}$

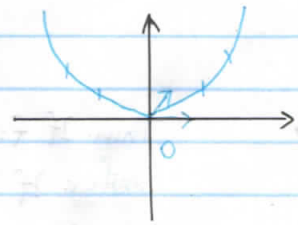
$f'(a;e_i)=\lim_{h\to0}h^{-1}(f(a+he_i)-f(a)) = \lim_{h\to0}h^{-1}\left(f\begin{pmatrix}a_1\\ a_2\\ a_i+h\\ \vdots\\ a_n\end{pmatrix}-f\begin{pmatrix}a_1\\ \vdots\\ a_i\\ \vdots\\ a_n\end{pmatrix}\right)$

$=\frac{d}{dx_i}$(the function you get from f by fixing $x_1\cdots x_n$ to be $a_1\cdots a_n$ except x_i)(a).

$=: \frac{\partial f}{\partial x_i}(a)$

* $\frac{\partial x^y}{\partial y}=x^y\log x$ $\quad (a^x)'=a^x\log a$
* $\frac{\partial x^y}{\partial x}=yx^{y-1}$ $\quad (f:\mathbb{R}^2_{x,y}\to\mathbb{R})$.

Example. $f(x,y)=\begin{cases}1 & y=x^2\ \&\ x\neq0.\\ 0 & \text{otherwise.}\end{cases}$ $\qquad r:\mathbb{R}^2\to\mathbb{R}$.



$f'(0;u)=0$. No claim rule.

Differential. $(f(a+h)-f(a))/h\overset{h\sim0}{\sim}f'(a)$.

$f(a+h)-f(a)\overset{h\sim0}{\sim}f'(a)\cdot h$. $\quad \underset{\mathbb{R}^n}{f(a+h)}\overset{h\sim0}{\sim}\underset{\mathbb{R}^m}{f(a)}+f'(a)\cdot \underset{\mathbb{R}^m}{h}$

make sence $f:\mathbb{R}^n\to\mathbb{R}^m$

$f'(a)$ should be a p.t. $\overset{M_{m\times n}}{\mathbb{R}^n\to\mathbb{R}^m}$

Oct 7.

Remarks $F'(a;u)=\lim_{h\to0}\frac{f(a+hu)-f(a)}{h}$ $\quad (f:\mathbb{R}^n\to\mathbb{R}^m)$

For $F:\mathbb{R}\to\mathbb{R}$, $f(a+h)\sim f(a)+f'(a)h$ $\quad f'(a)\ \mathbb{R}^n\to\mathbb{R}^m$, $F'(a)\in M_{m\times n}(\mathbb{R})$

$\lim_{h\to0}\frac{f(a+h)-f(a)}{h}=f'(a) \iff \lim_{h\to0}\frac{F(a+h)-f(a)-f'(a)h}{h}=0.$ $\quad \lim\frac{RHS-LHS}{h}=0$.

Def. $f:\mathbb{R}^n\to\mathbb{R}^m$ is diff. at $a\in\mathbb{R}^n$ if there exists a matrix $B\in M_{m\times n}(\mathbb{R})$ s.t. $f(a+h)\sim f(a)+B\cdot h$ for small $h\in\mathbb{R}^n$. ?

① 1. Book's way. $(\|f(a+h)-f(a)-B\cdot h\|/\|h\|)\xrightarrow{h\to0}0$.

↑ the type of norms doesn't matter

② 2. Dror's way. $o(h)=\{\phi:\mathbb{R}^n\to\mathbb{R}^m \mid \|\phi(h)\|/\|h\|\to0 \text{ as } h\to0.\}$ $f(a+h)-f(a)-Bh\in o(h)$

(loosely). $f(a+h)=f(a)+Bh+o(h)$

Thm 1. If B exists, it is unique. Call it $Df(a)$. "the differential of f at a"

Thm 2. If f is a constant, $(Df)(a)=0_{m\times n}$

Thm 3. If f is linear, $f(x)=Ax$ where $A\in M_{m\times n}$ $\quad (Df)(a)=A$

Thm 4. $D(cf)=cD(f)$. $D(f+g)=D(f)+D(g)$.

Thm 5. If f is diff. $f'(a;u)=Df(a)\cdot u$, so therefore $Df(a)=\left(f'(a;e_1)\,\middle|\,f'(a;e_2)\,\middle|\cdots\right)$

$$= \begin{pmatrix} \partial F_1/\partial x_1 & \partial F_1/\partial x_2 & \cdots & \partial F_1/\partial x_n \\ \partial F_2/\partial x_1 & \vdots & & \vdots \\ \vdots & & & \\ \partial F_m/\partial x_1 & \vdots & & \partial F_m/\partial x_n \end{pmatrix}$$ "Jacobian matrix of f"

$$\text{*. } f(x,y) = \begin{pmatrix} e^x \cos y \\ e^y \cos x \\ \frac{1}{1+x^2+y^2} \end{pmatrix} \quad \mathbb{R}^2 \to \mathbb{R}^3 \qquad \underset{3\times 2}{Df} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \\ \frac{-2x}{(1+x^2+y^2)^2} & \frac{-2y}{(1+x^2+y^2)^2} \end{pmatrix}$$

Pf. (of thm 1). Assume $f(a+h) = f(a) + B_1 h + o(h)$

$\underline{f(a+h) = f(a) + B_2 h + o(h).}$

$(B_1 - B_2)h + (\phi_1 - \phi_2)$ $\quad \phi_1, \phi_2 \in o(h)$. $o(h)$ is a v.s.

$\Rightarrow (B_1 - B_2)h \in o(h)$, i.e. $(B_1 - B_2)h/|h| \to 0$.

$B_1 - B_2 = 0$. $\therefore B_1 = B_2$ $\square$.

(of thm 2). $f(x) = C \in \mathbb{R}^m \ \forall x$ $\quad f(x+h) = f(x) + B\cdot h + o(h)$.

$C = C + 0h + 0 \quad \Rightarrow (Df)(a) = 0$.

(of thm3) $f(x) = Ax$. $f(a+h) = A(a+h) = A(a) + Ah = f(a) + Ah + 0$. $Df(a) = A$.

(of thm4) $(f+g)(a+h)$ --- ---.

(of thm5) $f'(a;u) = \lim_{h\to 0} \frac{f(a+h\cdot u) - f(a)}{h} = \lim_{h \to 0} \frac{f(a) + Df(a)\cdot h \cdot u + o(h\cdot u) - f(a)}{h} = Df(a)\cdot u$.