Dec 3, 2012

The Sturm Comparison Theorem

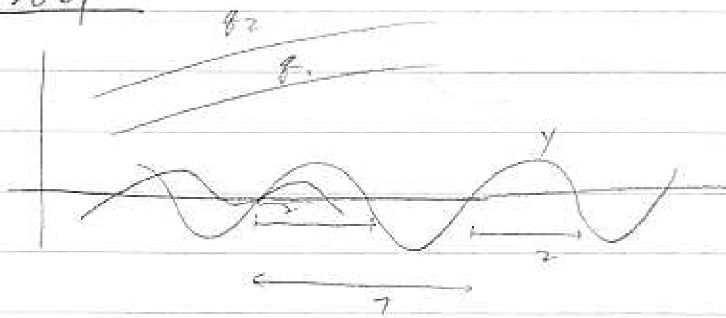
If $y_1'' + q_1 y_1 = 0$
$y_2'' + q_2 y_2 = 0$ & $q_2 > q_1$ on some interval $q_1, q_2 > 0$

Then strictly between any two zeros of y_1
there is a zero of y_2 [hence "y_2 oscillates more"]

Corollary

Assume $y'' + qy = 0$ & $q > 0$ & q is increasing.
Then the distance between successive zeros of y
is decreasing (q is decreasing, distance is increasing)

proof



example

Bessel of order 0 $x^2 y'' + xy + (x^2 - \cancel{\alpha^2}) y = 0$
by a change of variables $V = \sqrt{x}\, y$, equation becomes

$$V'' + \underbrace{\left(1 + \frac{1}{4x^2}\right)}_{Q} V = 0$$

distance between zeros

1. Q is $\downarrow$ therefore $dbz \uparrow$
2. $Q > 1$, so V oscillates faster than $z'' + z = 0$, $z = A\sin(\theta + x)$
   so $dbz < \pi$
3. $Q < 1.01$ eventually, so for large x

$$dbz(y) > dbz(z'' + 1.01 z = 0) = \frac{\pi}{\sqrt{1.01}}, \quad z = A\cos(\theta + \sqrt{1.01}\, x)$$

so $dbz(y)$ increases to π

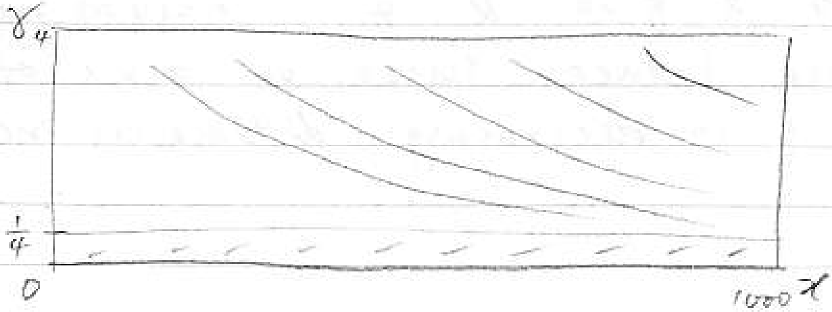
example

Euler's equation: $y'' + \frac{\gamma}{x^2} y = 0$

indicial : $\alpha(\alpha-1)+\gamma=0$

$\alpha^2-\alpha+\gamma=0$

$b^2-4ac=1-4\gamma$ $\begin{cases} \text{oscillate} & \gamma > \frac{1}{4} \\ \text{not oscillates} & \gamma \le \frac{1}{4} \end{cases}$



Corollary

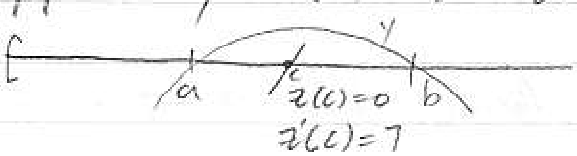
Suppose $y'' + q y = 0$

1. If $\exists \gamma > \frac{1}{4}$ s.t. $q \ge \frac{\gamma}{x^2}$ on $[A,\infty)$ then y oscillates

2. If $q \le \frac{1}{4x^2}$ on $[A,\infty)$ then y has at most one zero on $[A,\infty)$.

proof of (2)

Suppose y has 2 zeros in $[A,\infty)$



By sturm, any solution of $z'' + \frac{1}{4x^2} z = 0$ will have a zero on $[a,b]$. But $z'' + \frac{1}{4x^2} z$ has a solution with no zeros at all : $z=\sqrt{x}$

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Comment

Changing the independent variable : $\underset{\text{dep}}{y}(\underset{\text{indep.}}{x})$

$y'' + py' + qy = 0 \quad z = v(x)$

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} v'$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\Big(\underbrace{v' \frac{dy}{dz}}_{Y}\Big) = v''\frac{dy}{dz} + v' \frac{d}{dx}\Big(\underbrace{\frac{dy}{dz}}_{Y}\Big)$

$= v''\frac{dy}{dz} + (v')^2 \frac{d^2y}{dz^2}$

$v''\frac{dy}{dz} + (v')^2\frac{d^2y}{dz^2} + pv'\frac{dy}{dz} + qy = 0$

$(v')^2\frac{d^2y}{dz^2} + \underbrace{(v'' + pv')}\frac{dy}{dz} + qy = 0$

if $v'' + pv' = 0$. then

$\frac{d^2y}{dz^2} + \underbrace{\frac{q}{(v')^2}}_{Q} y = 0$

set $v' = \rho$

$\rho' + p\rho = 0$

$\rho = e^{-\int p} \rightsquigarrow v = \int e^{-\int p}$