Oct 5, 2012 Mat 267

HW3 Task 2 Problem 6 is postponed!

Read Along : The Gelfand and Fomin

Euler - Lagrange

y "minimizes" $J(y) = \int_a^b F(x, y, y')\,dx$ with $y(a) = A, y(b) = B$

$$\Rightarrow F_y - \frac{d}{dx} F_{y'} = 0$$

Second order ODE

$F = \mathbb{R}^3_{x,y,y'} \longrightarrow \mathbb{R}$

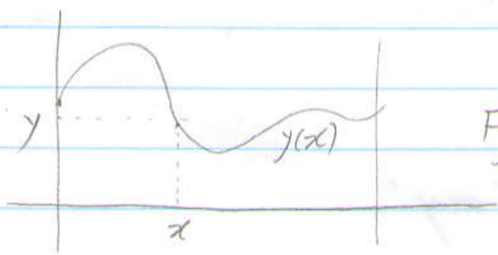
$F_{y'} : \mathbb{R}^3_{x,y,y'} \longrightarrow \mathbb{R}$

$F_{y'}(x, y(x), y'(x)) : \mathbb{R}_x \longrightarrow \mathbb{R}$

$F_2(x, y(x), y'(x)) - \frac{d}{dx} F_3(x, y(x), y'(x)) = 0$

Special Cases

1. F does not depend on y' : $F_{y'} = 0$



$F_y = 0$

E-L : $F_y = 0 \Rightarrow$ algebraic equation

2. F does not depend on y : $F_y = 0$

E-L : $\frac{d}{dx} F_{y'} = 0 \Rightarrow F_{y'} = C$ 1st order ODE "easy"

example

$J(y) = \int [\frac{1}{2} m (y')^2 - V(y)]\,dx$, $F = \frac{1}{2} m (y'^2) - V(y)$

$F_y = 0 \Leftrightarrow V = C$ (WLOG $V = 0$)

$F_{y'} = m \cdot y' = C$

"in the absence of force, momentum is conserved"

Case 3

F does not depend on x: $F_x = 0$

$\Rightarrow$ "Conservation of Energy"

Euler - Lagrange

$$0 = F_y - \frac{d}{dx} F_{y'} = F_y - 1\cdot 0 - F_{y'y}\cdot y' - F_{y'y'}\cdot y'' \quad (*)$$

$$(*) \times y' \Rightarrow 0 = \underline{y' F_y - (y')^2 F_{y'y} - F_{y'y'} y' y''}$$

$$= \frac{d}{dx}(F - y' F_{y'}) \quad \text{same}$$

$$\left(= y' F_y + \cancel{y'' F_{y'}} - \cancel{y'' F_{y'}} - y'(F_{y'y} y' + F_{y'y'} y'') \right.$$

$$\left. = \underline{y' F_y - (y')^2 F_{y'y} - F_{y'y'} y' y''} \right)$$

$\Rightarrow F - y' F_{y'} = c$ "1st order ODE!"

example 1

$$F = \frac{1}{2} m (y'^2) - V(y)$$

$$F - y' F_{y'} = \frac{1}{2} m y'^2 - V(y) - y' m y'$$

$$= -\left(\frac{1}{2} m y'^2 + V(y)\right)$$

$\Rightarrow$ Conservation of Minus the Energy

example 2 (Brachistochrone)

$$F = \sqrt{\frac{1+y'^2}{y}} \Rightarrow c = \sqrt{\frac{1+y'^2}{y}} - y' \frac{y'}{\sqrt{y}\sqrt{1+y'^2}}$$

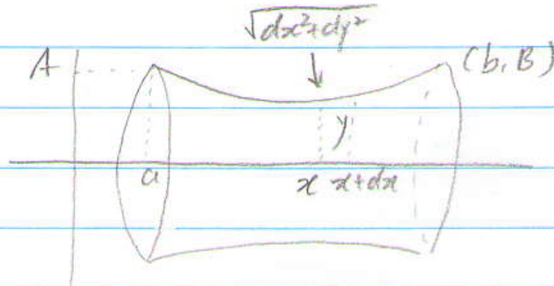
$$= \frac{1+y'^2 - y'^2}{\sqrt{y(1+y'^2)}}$$

$$= \frac{1}{\sqrt{y(1+y'^2)}}$$

$$\cdots \quad y' = \sqrt{\frac{d-y}{y}}$$

example 3

"Find the rotational loaf of bread having the least crust



$$A = \int 2\pi y \cdot \sqrt{1+y'^2}\,dx$$

Minimize surface area

$$\text{Area} = \int F, \quad F = y\sqrt{1+y'^2}$$

Conservation of Energy.

$$y\sqrt{1+y'^2} - y'\frac{yy'}{\sqrt{1+y'^2}} = c$$

$$\cdots \; y' = \sqrt{\left(\frac{y}{c}\right)^2 - 1} \qquad u' = \sqrt{1-u^2}$$

$$\Rightarrow y = c \cdot \cosh\frac{x-c'}{c}$$