TT2 : Much like TT1.

Def. $h: \mathbb{R}^n \to \mathbb{R}^n$ isometry $\Leftrightarrow \forall x,y \;\; d(h(x),h(y)) = d(x,y)$.

Thm. h is an isometry iff it is of the form $h(x) = P + Ax$, where $P \in \mathbb{R}^n$ & $A \in M_{n\times n}$ s.t. $A^TA = 1$.

comments 1. Such h is "volume preserving"

2. A l.t. A s.t. $A^TA = 1$ is called "orthogonal"

this means $\left(V_1 \middle| V_2 \middle| \cdots \middle| V_n\right) = A \quad A^TA = 1 \Leftrightarrow \langle V_i, V_j \rangle = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$\Rightarrow V_i \perp V_j$ for $i \neq j$. & $\|V_i\| = 1 \;\; \forall i$

$\Rightarrow \{V_i\}$ forms an orthonormal basis.

~~A mo~~ A maps the std basis to an orthonormal basis. A is a generalized rotation.

3. Rotation matrices / orthogonal matrices

$O(n) = \{A \in M_{n\times n}(\mathbb{R}) : A^TA = 1\}$.

form a group: 1: $A, B \in O(n) \quad A \cdot B \in O(n) \quad A(BC) = (AB)C$

2: $\exists\, I \in O(n)$ s.t. $A1 = 1A \quad \forall A \in O(n)$

3. $A \in O(n) \;\; \exists B \in O(n)$ s.t. $AB = BA = 1$.

Pf 1. If A & B satisfy $A^TA = 1 = B^TB$, does $AB \in O(n)$?

$(AB)^T AB = B^TA^TAB = B^T 1 B = B^TB = 1$.

2. Take $1 = 1_{n\times n}$ $\;1 \in O(n)$? $\quad 1^T1 = 1 \cdot 1 = 1^2 = 1$.

3. $A^TA = 1 \Leftrightarrow A^T = A^{-1}$ Is $A^T \in O(n)$? Ans $(A^T)^TA^T = A \cdot A^T$

"For a square matrix, a left inverse is also a right inverse" $\Rightarrow AA^T = 1$.

proof (of thm above) ($\Leftarrow$) given $h(x) = P + Ax$, $A \in O(n)$. $\quad d(h(x), h(y)) = \|h(x) - h(y)\| = \|P + Ax - (P + Ay)\|$

$= \|A(x-y)\| = \sqrt{\langle A(x-y), A(x-y)\rangle}$

$d(h(x),h(y)) = ((A(x-y))^TA(x-y))^{1/2} = [(x-y)^T A^TA(x-y)]^{1/2} = \langle x-y, x-y\rangle^{1/2} = \|x-y\| = d(x,y)$

($\Rightarrow$) 1. WLOG, $h(0) = 0 \quad (P = 0)$

Indeed, h is iso iff $h_1(x) := h(x) - \underbrace{h(0)}_{P}$ is iso. Yet $h_1(0) = 0$

So $h_1(x) = Ax$ for $A \in O(n)$. So $h(x) = \overbrace{h(0)}^{P} + Ax$.

2. h "preserves norms" $\|x\| = d(x,0) = d(h(x), h(0)) = d(h(x), 0) = \|h(x)\|$

3. h preserves inner product $\langle h(x), h(y) \rangle = \langle x, y \rangle$?

$\underset{d(x,y)^2}{\|x-y\|^2} = \langle x-y, x-y \rangle = \langle x,x \rangle - \langle x,y \rangle - \langle y,x \rangle + \langle y,y \rangle$

$d(h(x),h(y))^2 \qquad = \|x\|^2 - 2\langle x,y \rangle + \|y\|^2$

$\|h(x) - h(y)\|^2 = \|h(x)\|^2 - 2\langle h(x), h(y) \rangle + \|h(y)\|^2$

$0 = 0 - 2\langle x,y \rangle + 2\langle h(x), h(y) \rangle + 0. \quad \Rightarrow \langle h(x), h(y) \rangle = \langle x,y \rangle$.

4. Set $A = \left(h(e_1) \middle| h(e_2) \middle| \cdots \middle| h(e_n)\right)$

Claim: $A \in O(n)$. $\quad (A^TA)_{ij} = \langle h(e_i), h(e_j) \rangle = \langle e_i, e_j \rangle = \delta_{ij} \Rightarrow A^TA = 1$.

Jan 11.

TT: Tue Jan 17. 5-7 pm.

Def. $h: \mathbb{R}^n \to \mathbb{R}^n$ is an "isometry" if $\forall x,y$ $d(h(x), h(y)) = d(x,y)$ (Euc)

Thm. h is an isometry iff it is of the form $h(x) = P + Ax$, where $A \in M_{n\times n}$ satisfies $A^TA = I$.

Already know: WLOG, $h(0)=0$; h preserves norms & dot products. $A := (h(e_1)|h(e_2)|\cdots|h(e_n)) \in O(n)$
$[A^TA = A]$

Claim: $h(\sum x_ie_i) = \sum x_ih(e_i)$

If true, $h\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = h(\sum x_ie_i) \overset{\text{claim}}{=} \sum x_ih_i(e_i) = A\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = Ax$

pf of claim: Let $\Delta = h(\sum x_ie_i) - \sum x_ih(e_i)$

$\langle \Delta, h(e_j)\rangle = \langle h(\sum x_ie_i), h(e_j)\rangle - \sum x_i\langle h(e_i), h(e_j)\rangle$

$= \langle \sum x_ie_i, e_j\rangle - \sum x_i\langle e_i, e_j\rangle = 0.$

But $h(e_j) = Ae_j$, so $0 = \langle \Delta, h(e_j)\rangle = \langle \Delta, Ae_j\rangle = \Delta^TAe_j\ \forall j \Rightarrow \Delta^TA = 0.$

But A is invertible, so $\Delta^T = 0$ so $\Delta = 0$. □

Gram-Schimt process,

If $\{u_i\}$ is a basis of an inner product space (for this class, okay to think $V = \mathbb{R}^n$, $\langle a,b\rangle = a^Tb$)

Then there exists (almost unique) orthonormal basis $\{v_i\}$ s.t. $\forall k$, $1 \le k \le n = \dim V$.
$\operatorname{span}(u_i)_{i=1}^k = \operatorname{span}(v_i)_{i=1}^k$

Example. $u_1 = \begin{pmatrix}3\\4\end{pmatrix}$ & $u_2 = \begin{pmatrix}1\\1\end{pmatrix}$ in $\mathbb{R}^2$ $\quad v_1 = \pm\frac{u_1}{\|u_1\|} = +\begin{pmatrix}3\\4\end{pmatrix}/5 = \begin{pmatrix}3/5\\4/5\end{pmatrix}$

$v_2' = u_2 - \langle u_2, v_1\rangle v_1 = \begin{pmatrix}1\\1\end{pmatrix} - \frac{7}{5}\begin{pmatrix}3/5\\4/5\end{pmatrix} = \begin{pmatrix}4/25\\-3/25\end{pmatrix}$

$v_2 = \pm\frac{v_2'}{\|v_2'\|} = \begin{pmatrix}4/25\\-3/25\end{pmatrix}/(1/5) = \begin{pmatrix}4/5\\-3/5\end{pmatrix}$

Now in general. $v_1' = u_1 \qquad v_1 = \pm v_1'/\|v_1'\|$

$v_2' = u_2 - \langle u_2, v_1\rangle v_1 \qquad v_2 = \pm v_2'/\|v_2'\|$

$v_3' = u_3 - \langle u_3, v_1\rangle v_1 - \langle u_3, v_2\rangle v_2 \qquad v_3 = \pm v_3'/\|v_3'\|$

$v_k' = u_k - \sum_{j=1}^{k-1}\langle u_k, v_j\rangle v_j \qquad v_k = v_k'/\|v_k'\|$

Claim: The process works. 1: v_i are O.N. 2. $\operatorname{span}(u_i)_{i=1}^k = \operatorname{span}(v_i)_{i=1}^k$

proof: Exercise / MAT 247.

K-dim Volumes in $\mathbb{R}^3$.

Q: Given $v_1 \cdots v_k$ in $\mathbb{R}^n$ what's vol (parallogram spanned by these) $= V(v_1, \cdots v_k)$

Want: 1. If $A^TA = I$, $A \in M_{n\times n}(\mathbb{R})$. $V(v_1, \cdots v_k) = V(Av_1, \cdots Av_k)$

2. If $v_1 \cdots v_k \subset \mathbb{R}^k \times \{0_{n-k}\} \subset \mathbb{R}^n$ then $v_i = \begin{pmatrix}y_i\\0\\ \vdots\\0\end{pmatrix}\}_{n-k}$ & $V(v_1 \cdots v_k) = |\det(y_1|y_2|\cdots|y_k)|$

Thm. V exists and is unique.

Jan 13.

Thm. There is a unique $V: (\mathbb{R}^n)^k \to \mathbb{R}_{\geq 0}$ s.t.

1: If $h: \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal trans, & $x_i \in \mathbb{R}$. $V(h(x_1), \dots h(x_k)) = V(x_1, \dots, x_k)$.

2. If $x_i \in \mathbb{R}^k \times \{0\}$. so $x_i = \binom{y_i}{0}$ with $y_i \in \mathbb{R}^k$, then $V(x_1, \dots x_k) = |\det(y_1) \cdots (y_k)|$

Further more: 3. $V(x_1 \dots x_k) = 0 \iff \{x_i\}$ is dependent.

4. If $X = (x_1 | \dots | x_k) \in M_{n \times k}$ then $V(x_1 \dots x_k) = |\det(X^T \cdot X)|^{1/2}$ (5).

pf. We want to show that 1-2 above determine $V(x_1, \dots, x_k)$

Let $W = \text{span}(x_1, \dots, x_k)$. Find an O.N basis $(f_i)_{i=1}^{l}$ of W where $l < k$.

Extend to a ON basis $(f_i)_{i=1}^{n}$ of $\mathbb{R}^n$, $f_1 \to e_1 \quad \cdots \quad f_l \to e_l \quad f_n \to e_n$

Let $A = (f_1 | f_2 | \cdots | f_n)$ so $Ae_i = f_i$

A is orthogonal, so its invertible with orthogonal inverse.

Let h be the linear transformation represented by A^{-1}, it is orthogonal and $h(f_i) = e_i$

So $h(f_1) \dots h(f_l) \in \mathbb{R}^l \subset \mathbb{R}^k$

Now $V(x_1, \dots x_k) \overset{1}{=} V(h(x_1) \cdots h(x_k))$ but each k: $x_i \in W$, & $h(f_i)_{i=1}^{l} \in \mathbb{R}^k \Rightarrow h(W) \subset \mathbb{R}^k$

So $h(x_i) \in \mathbb{R}^k$, and the RHS is determined by (2).

For existence, n.t.s. (5) $\Rightarrow$ 1, 2

1: Suppose h is orthogonal, meaning $h(x) = Ax$, where $A^T A = I$.

$V(h(x_1) \dots h(x_k)) = V(Ax_1, \dots Ax_k) = |\det(X_h^T X_h)|^{1/2} = |\det(X^T A^T A X)|^{1/2} = |\det(X^T X)|^{1/2} = V(x_1 \cdots x_k)$.

$X_h = (Ax_1 | \cdots | Ax_k) = A \cdot (x_1 \cdots x_k) = AX$

2. Suppose $x_i = \begin{pmatrix} y_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix}$ $\quad X = (x_1 | \cdots | x_n) = \left(\begin{array}{c|c|c|c} y_1 & y_2 & \cdots & y_k \\ 0 & 0 & & 0 \end{array} \right) = \left(\begin{array}{c} y_1 | \cdots | y_k \\ \hline 0 \end{array} \right) \begin{matrix} \}k \\ \}n-k \end{matrix} = \binom{Y}{0}$

$V(x_1 \cdots x_k) = |\det(X^T X)|^{1/2} = \left|\det\left((Y^T\ 0)\binom{Y}{0}\right)\right|^{1/2} = |\det(Y^T Y)|^{1/2} = |\det(Y)^2|^{1/2} = |\det(Y)|$

pf 3. ($\Leftarrow$) $\{x_i\}$ dep $\Rightarrow \exists\, a \neq 0$ s.t. $X \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} = 0 \Rightarrow X^T X a = 0 \Rightarrow X^T X$ is not invertible.

$\Rightarrow V = |\det(X^T X)|^{1/2} = 0$.

($\Rightarrow$). $V(x_1 \cdots x_k) = 0 \Rightarrow \det(X^T X) = 0 \Rightarrow \exists\, a \neq 0$ s.t $X^T X a = 0$, $\Rightarrow a^T X^T X a = 0$.

$(Xa)^T X a = 0 \Rightarrow \|Xa\|^2 = 0. \Rightarrow Xa = 0$ So col's of X are dep.

Example. $2 \subset 3$. $\quad x, y \in \mathbb{R}^3 \quad V(x,y) = |\det X^T X|^{1/2} = \left|\det\begin{pmatrix} \|x\|^2 & \langle x,y \rangle \\ \langle y,x \rangle & \|y\|^2 \end{pmatrix}\right|^{1/2}$

$X = (x | y) \quad X^T = \left(\frac{x}{y}\right) \quad = \left| \|x\|^2 \|y\|^2 - \langle x,y \rangle^2 \right|^{1/2} =$ Ans.

Ans $= \left| \|x\|^2 \|y\|^2 (1 - \cos^2\theta) \right|^{1/2} = \|x\| \|y\| |\sin\theta|$