Def: A sp X is called "cpt" if whenever you cover X with open sets, finitely many of these already cover X

$$\left(X = \bigcup_{\alpha \in I} U_\alpha,\ U_\alpha \text{ open}\right) \Rightarrow \left(\exists F \subset I \text{ finite set},\ X = \bigcup_{\alpha \in F} U_\alpha\right)$$

e.g.: $\mathbb{R}$ is not cpt

Indeed $X = \bigcup_{n>0} (-n, n)$

Yet no finite subcover

e.g.: A finite set is cpt

Thm: $[0,1]$ is cpt

$(0,1)$ is not cpt.

Indeed $(0,1) = \bigcup_{n} (\frac{1}{n}, 1)$

Thm: A cts fct on a cpt sp is bdd, meaning $\exists M$ s.t. $\forall x \in X$ $|f(x)| < M$.

Def: $A \subset X$ cpt if whenever you cover A with open sets, open in X? open in A?, you can find a finite subcover meaning finitely many of these open sets already cover A.

Claim: The two interpretations are the same.

pf: key pt: $V \subset A$ open iff $\exists U \subset X$ s.t. $U \cap A = V$

Thm: $[0,1]$ cpt

pf: Let U_α be open set s.t. $\bigcup U_\alpha = I = [0,1]$

Let $G = \{ y \in [0,1] \mid$ the interval $[0,y]$ can be covered by fin. many of the U_α's $\}$

Then $0 \in G$ because $[0,0] = \{0\}$

$\exists \alpha_0$ s.t. $0 \in U_{\alpha_0}$ and then $[0,0] \subset U_{\alpha_0}$

So $[0,0]$ has a fin. subcover.

G is bdd and nonempty. So $g = \sup G$ exists.

So $0 \le g \le 1$.

Claim: $g \in G$.

pf: As $g \in \bigcup U_\alpha$, can find some α_0 s.t. $g \in U_{\alpha_0}$

U_{α_0} open so $\exists \varepsilon > 0$ s.t. $(g-\varepsilon, g+\varepsilon) \subset U_{\alpha_0}$

As $g = \sup G$, $\exists y \in G$ s.t. $g - \varepsilon < y \le g$

but then by def of G, $\exists$ fin. many $U_{\alpha_1}, \dots, U_{\alpha_n}$

s.t. $[0,y] \subset \bigcup_{i=1}^{n} U_{\alpha_i}$.

Then $\bigcup_{i=0}^{n} U_{\alpha_i} \supset (g-\varepsilon, g+\varepsilon) \cup [0,y]$

$\supset [0,g]$

So $[0,g]$ has a fin. cover by U_α's. So $g \in G$

<u>Claim</u>: $g > 0$

pf: Some U_{α_0} covers 0.

It also covers $[0, \varepsilon)$ for some $\varepsilon > 0$

$\Rightarrow \sup G \geq \varepsilon$
> 0

<u>Claim</u>: $g = 1$ and we are done.

pf: if $g \neq 1$, then $g < 1$

Some U_{α_0} covers g. Some $U_{\alpha_1}, \cdots, U_{\alpha_n}$ cover $[0, g]$

Since U_{α_0} open, it covers $(g-\varepsilon, g+\varepsilon)$ for some $\varepsilon > 0$.

$$\bigcup_{i=0}^{n} U_{\alpha_i} \supset [0, g+\varepsilon)$$
$$\supset [0, g + \varepsilon/2]$$

So $g + \varepsilon/2 \in G$ $(\Rightarrow\Leftarrow)$

<u>Thm</u>: A subset $A \subset \mathbb{R}^n$ cpt $\Rightarrow$ A closed and bdd.

pf: $A \subset \bigcup_n B(0, n)$
$= \mathbb{R}^n$

if A cpt, then $\exists N$ s.t. $A \subset \bigcup_{n=1}^{N} B(0,n) = \bigcup (0, N)$

So A bdd.