Def: $A \subset X$ "cpt" means

$$\left(A = \bigcup_{\alpha\in I} U_\alpha,\ U_\alpha \text{ open in } A\right) \Rightarrow \left(\exists F \subset I \text{ finite s.t. } A = \bigcup_{\alpha\in F} U_\alpha\right)$$

"U_α's cover A"

$$\Leftrightarrow \left(A \subset \bigcup_{\alpha\in I} V_\alpha,\ V_\alpha \text{ open in } X\right) \Rightarrow \left(\exists F \subset I \text{ finite s.t. } A \subset \bigcup_{\alpha\in F} V_\alpha\right)$$

Thm: $A \subset \mathbb{R}^n$ cpt iff it is closed and bdd.

$\exists M\ \forall x\in X,\ \|x\|<M$

$\Leftrightarrow \exists N\ \forall x\in X,\ |x|<N$

pf: $(\Rightarrow)$ Sps $X \subset \mathbb{R}^n$ cpt.

X bdd was done last time.

① $X \subset \bigcup_{k=1}^{\infty} B(0,k) = \mathbb{R}^n \Rightarrow X \subset \bigcup_{k=1}^{N} B(0,k) = B(0,N)$

② $\|x\|$ cts, so it is bdd

X closed:

Let $x \notin X$.

For any k, consider $D_k = \{y \mid d(x,y) > \frac{1}{k}\}$.

Easy to show that D_k open.

$$\bigcup_{k=1}^{\infty} D_k = \mathbb{R}^n \setminus \{x\} \supset X$$

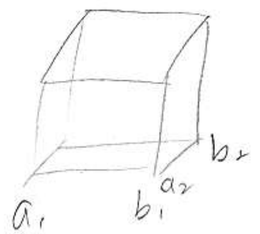
So by cptness, $\exists N$ s.t. $D_N = \bigcup_{k=1}^{N} D_k \supset X$

So $X \subset D_N$

So $\mathbb{R}^n \setminus X \supset \mathbb{R}^n \setminus D_N \supset B(x, \frac{1}{N})$

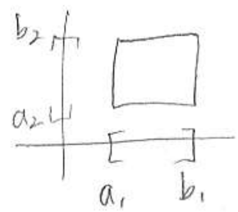
So X^c open and X closed

($\Leftarrow$)



cpt $[a_1, b_1] \times [a_2 \times b_2] \times [a_3 \times b_3]$

Q: If X & Y cpt, is $X \times Y$ cpt?

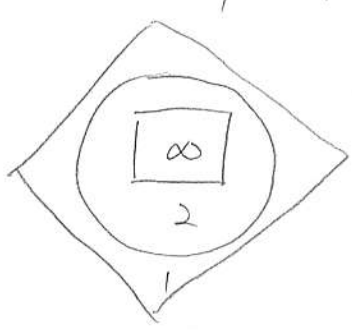


Def: Sps (X, d_1) & (Y, d_2) metric

Define on $X \times Y$,

$$d((x,y),(x',y')) = d_1(x,x') + d_2(y,y') \qquad 1$$
$$\text{or } \sqrt{d_1(x,x')^2 + d_2(y,y')^2} \qquad 2$$
$$\text{or } \max(d_1(x,x'), d_2(y,y')) \qquad \infty$$

Claim: All three possibilities define a metric and open sets rel. any of the options are the same



open balls

$$B_\infty((x,y),\varepsilon) = \{(x',y') \mid d_\infty((x,y),(x',y')) < \varepsilon\}$$

$\cap$ $X \times Y$

$$\max(d_1(x,x'), d_2(y,y')) < \varepsilon$$

$$= \{(x',y') \mid d_1(x,x') < \varepsilon,\ d_2(y,y') < \varepsilon\}$$

"an open square around (x,y)"

Thm: If X & Y cpt, so is $X \times Y$

pf: Let $\{W_\alpha\}$ be an open cover of $X \times Y$

Lem: WLOG, each W_α is of the form

$W_\alpha = U_\alpha \times V_\alpha$ where U_α open in X & V_α open in Y

Indeed, each W_α is a union of squares, so consider the cover of $X \times Y$ by all these squares. If I find a finite subcover using these squares it clearly defines a finite subcover using the original W_α's

Claim: If $X \times Y$ is covered by $U_\alpha \times V_\alpha$, then for every $x_0 \in X$, we can find $\varepsilon > 0$ s.t. $B(x_0, \varepsilon) \times Y$ is covered by finitely many of the $U_\alpha \times V_\alpha$'s

pf: By compactness of Y, $\exists F$ finite s.t.

$$\bigcup_{\alpha \in F} U_\alpha \times V_\alpha \supset \{x_0\} \times Y$$

But
$$\bigcup_{\alpha \in F} U_\alpha \times V_\alpha \supset \bigcup_{\alpha \in F} \left(\bigcap_{\beta \in F} U_\beta \right) \times V_\alpha$$
$$\supset \left(\bigcap_{\beta \in F} U_\beta \right) \times Y$$