24th Fri March Hour 0.67
Read ~~also~~ along 37-38

Practical def for $\int_M w$: chop $\dot{p}$ to pieces w/ measure-0 exceptions.

formal def for $\int_M w$: Find PO1 subordinate to positive charts of M * $\phi_i \in C^\infty$.

* Supp $\phi_i \subset$ im(pos. chart α_i) of locally finite

* $\sum \phi_i = 1$. Then set

$$ {}^{\phi}\!\int_M w = \sum_{i\in I}\int_M \phi_i \cdot w = \sum_{j\in J}\int_M \psi_j \cdot w = {}^{\psi}\!\int_M w := \int_M w $$

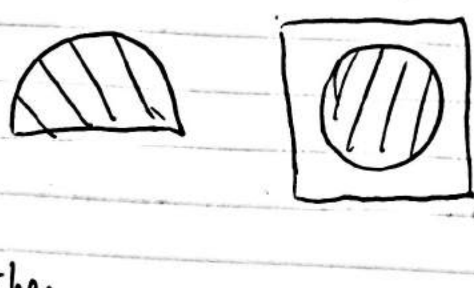
Thm "Stoke's": If M^k is compact and orinted,

and $w \in \Omega^{k-1}(M)$,

then $\int_M dw = \int_{\partial M} w$

Proof (trivial): Case I. Supp $w \subset \mathrm{Im}\,\alpha$,

where $\alpha : \underset{\cap\, \mathbb{R}^k}{Q} \xrightarrow{\text{positive}} M$, $\lambda = \alpha^* w, \in \Omega^{k-1}(Q)$



~~Supp α^*~~ w

Supp $\alpha^* w \subset$ int Q

Then,

Assume $\lambda = \sum_{i=1}^{k} \lambda^i \, dx1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dxk$,

$$\int_M dw = \int_Q \alpha^* dw = \int_Q d\lambda = \sum_{i=1}^{k} (-1)^{i-1} \int_Q \frac{\partial \lambda^i}{\partial x_i} = \int_{\partial M} w \quad ①$$

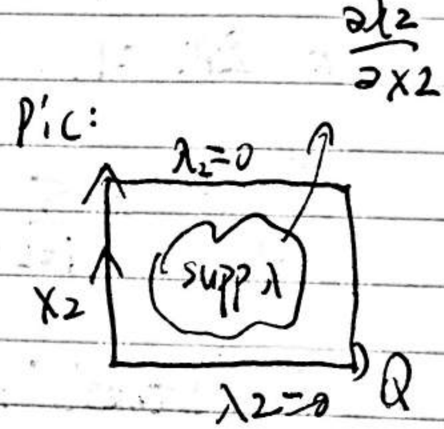
$$= 0 \quad ②$$

① : True because supp $w \cap \partial M = \phi$

② : True by fundemental theorem of calc, using

$\lambda|_{\partial Q} = 0$

Pic:



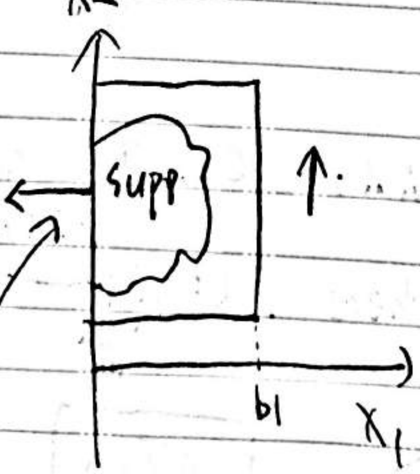
$$d\lambda = \sum_i \frac{\partial \lambda^i}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dxk$$
$$= \sum_i^{k} (-1)^{i-1} \frac{\partial \lambda^i}{\partial x_i} dx\underline{k}$$

Case 2: (boundry charts) : $\lambda \in \Omega^{k-1}(\mathbb{R}^k_{x_1 \geq 0} = H_1^k)$

Support $\lambda \subset \mathrm{int}_{H_1^k} Q$

$$Q = [0, b_1] \times \underbrace{\prod_{i=2}^{k} [a_i, b_i]}_{Q'}$$

$$= [0, b_1] \times Q'$$



Outward pointing normal, it's − if evaluate

$$\int_M dw = \int_Q d\lambda = \sum_{i=1}^{k} (-1)^{i-1} \int_Q \frac{\partial \lambda_i}{\partial x_i}$$

$$= \int_Q \frac{\partial \lambda_1}{\partial x_1}$$

fubini

$$= \int_{Q'} \int_{x_1 \in [0, b_1]}^{b_1} \frac{\partial \lambda_1}{\partial x_2}$$

$$= \int_{Q'} \lambda_1 \Big|_0^{b_1}$$

$$= -\int_{x' \in Q'} \lambda_1 (0, x')$$

$$= -\int_{\{0\} \times Q'} \lambda_1 \, dx_2 \wedge \cdots \wedge dx_k$$

$$= -\int_{\{0\} \times Q'} \lambda = \int_{\partial Q} \lambda = \int_{\partial M} W$$

Case 3 $W = \cancel{\Sigma} \, \Sigma \phi_i w.$ $\quad \int_M dw \overset{?}{=} \int_{\partial M} w$

linearity : multiply function? $d\phi_i w \neq \phi_i dw$!

$$\int_{\partial M} w = \int_{\partial M} (\Sigma \phi_i) w = \Sigma \int_{\partial M} \phi_i w$$

$$= \sum_i \left(\int_M d(\phi_i w) \right)$$

ϕ_i : zero-form

$$= \sum_i \int_M (d\phi_i \wedge w + \phi_i \wedge dw)$$

$$\sum_i \int_M d\phi_i \wedge w = \int_M \sum_i (d\phi_i \wedge w)$$

$$= \int_M (\sum_i d\phi_i) \wedge w$$

$$= \int_M d(\overbrace{\Sigma \phi_i}^{\to 1}) \wedge w$$

$$= \int_M d_1 \wedge w = 0$$

$$= \sum_i \int_M \phi_i \wedge dw$$

$$= \int_M dw$$

Example : $M = [0,1] \subset \mathbb{R}^1$, $w = F$

$\partial M = \{0, 1\}$, conclusion: From Hw18

1. An oriented point is $(\pm, p)$
2. $\int F = S \cdot F(p)$

S = sign $\leftarrow \{s, p\}$ p = point $\in \mathbb{R}^n$

3. $\partial [0,1] = \{+, 1\} \cup \{-, 0\}$

$$\boxed{dw = \frac{\partial F}{\partial x} dx}$$

$$\int_0^1 \overbrace{\frac{\partial F}{\partial x}}^{= F'} = \int_{[0,1]} \frac{\partial F}{\partial x} dx = \int_M dw = \int_{\partial M} w = \int_{\{+,1\} \cup \{-,0\}} F$$

$$= F(1) + -F(0)$$