MAT 257 · Revision of perivous materials. Sep 12. Sep 14.

Def. V is a vector space if. $\forall \cdot x, y, z \in V$; c, d scalar

i) $x+y=y+x$ ii) $x+(y+z)=(x+y)+z$. iii) there is a unique vector $0 \in V$

iv) $x+(-1)x=0$ v) $1x=x$ vi) $c(dx)=(cd)x$ vii) $(c+d)x=cx+dx$ viii) $c(x+y)=cx+cy$

*Addition & multiplication (with scalar):

$x, y \in V$ then $x+y=(x_1+y_1, x_2+y_2, \cdots, x_n+y_n)$

$x \in V$, c scalar, $cx=(cx_1, \cdots, cx_n)$

Def. Let V be a vector space. A set $a_1 \cdots a_m$ of vectors in V is said to $\boxed{\text{span}}$ V if $\forall x \in V$, $\exists$ scalars $c_1 \cdots c_m$ s.t $x=\sum_{i=1}^{m} c_i a_i$.

- In this case, we say that x can be written as a $\boxed{\text{linear combination}}$ of the vectors $a_1 \cdots a_m$.
- The set of vectors $a_1 \cdots a_m$ is said to be $\boxed{\text{independent}}$ if to each x in V there exist at most one m-tuple of scalars $c_1 \cdots c_m$ such that $x=c_1a_1+\cdots+c_m a_m$.

  Equivalently, $\sum_{i=1}^{m} d_i m_i = 0 \iff d_1=\cdots=d_m=0$

Thm 1.1. Suppose V has a basis consisting of m vectors. Then any set of vectors that spans V has at least m vectors; any set of vectors of V that is independent has at most m vectors.

In particular, any basis for V has exactly m vectors.

Def. If V has a basis consisting of m vectors, we say that m is the $\boxed{\text{dimension}}$ of V. The vectorspace consisting of the zero vector alone has dimension zero.

e.g. the $\boxed{\text{standard basis}}$ for $\mathbb{R}^n$ is: $e_1=(1,0,0,\cdots,0)$ $e_2=(0,1,0,\cdots,0)$ $\cdots$ $\cdots$ $e_n=(0,0,0,\cdots,1)$

Thm 1.2. Let V be a vectors space of dimension m. If W is a linear subspace of V (different to V), then dim W < m.

Futhermore, any basis $a_1, \cdots a_k$ for W may be extended to a basis $a_1, \cdots a_k, a_{k+1}, \cdots a_m$ for V.

Def. $\langle \cdot, \cdot \rangle$ is an $\boxed{\text{inner product}}$ for vector space V if: $\forall x, y, z \in V$, c scalar,

i) $\langle x, y\rangle = \langle y, x\rangle$ ii) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

iii) $\langle cx, y\rangle = c\langle x, y\rangle = \langle x, cy\rangle$ iv) $\langle x, x\rangle > 0$ if $x \neq 0$.

e.g i) $\langle x, y\rangle = x_1y_1+\cdots+x_ny_n$ ii) $\|x\| = \langle x, x\rangle^{1/2}$

Def. $\|\cdot\|$ denotes a norm in V if ~~s.t.~~ $\forall x \in V$

i) $\|x\| > 0$ if $x \neq 0$ ii) $\|cx\| = |c|\,\|x\|$ iii) $\|x+y\| \leq \|x\| + \|y\|$. ~~triangle ineq.~~

iv) $\|x-y\| \geq \|x\| - \|y\|$

e.g. the sup norm. $|x|_{sup} = \max\{|x_1|, \cdots, |x_n|\}$.

Remark: sup S where S is a set, is the smallest number that larger than any ~~other~~ elements in S. superimum can be something outside of S if S has infinitely many elements. But maximum must be in the set.

* Sup norm for a matrix: $A = (a_{ij})$ then $|A|_{sup} := \max |a_{ij}|$

* In $\mathbb{R}^n$, $|x|_{sup} \leq \|x\| \leq \sqrt{n}\,|x|$

proof: In $\mathbb{R}^n$, $|x|_{sup} = x_i$ where $1 \leq i \leq n$

$$|x|_{sup} = |x_i| = \sqrt{x_i^2} \leq \sqrt{\sum_{j=1}^{n} x_j^2} = \|x\| \leq \sqrt{\sum_{j=1}^{n} x_i^2} = \sqrt{n x_i^2} = \sqrt{n}|x_i| = \sqrt{n}\,|x|_{sup}.$$

Def. $\boxed{\text{Matrix multiplication}}$: If A is a matix of size $n \times m$, B is a matrix of size $m \times p$. then the $\boxed{\text{product}}$ $A \cdot B$ is defined to be the matrix C of size $n \times p$ where:

$$C_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}.$$

(How to remember: $A \begin{pmatrix} a \text{---} a \end{pmatrix} \; B \begin{pmatrix} b \\ \vdots \\ b \end{pmatrix} \; \begin{pmatrix} x \end{pmatrix}$ x = sum of resultes of $a \cdot b$)

* Rules of Matrix multiplication: i) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ ii) $A \cdot (B+C) = A \cdot B + A \cdot C$

iii) $(A+B) \cdot C = A \cdot C + B \cdot C$ iv) $(cA) \cdot B = c(A \cdot B) = A \cdot (cB)$

Note: $AB \neq BA$!!!

Def. Identity matrix: $\forall k \in \mathbb{N}^*$, $\exists$ k by k matrix I_k s.t. if A is any n by m matrix. $I_n \cdot A = A$ and $A \cdot I_m = A$. i.e. $I_k = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

Thm 1.3. If A has size n by m, B has size m by p, then $|A \cdot B| \leq m|A||B|$

proof: Let $C = A \cdot B$

$$|c_{ij}| = \left|\sum_{k=1}^{m} a_{ik} b_{kj}\right| \leq \sum_{k=1}^{n} |a_{ik}||b_{kj}| \leq \sum_{k=1}^{m} |A| \cdot |B| = m|A| \cdot |B|$$

Def: Let V, W be vector spaces. a $\boxed{\text{linear transformation}}$ $T: V \to W$ is a mapping s.t. i) $T(x+y) = T(x) + T(y)$ $\forall x, y \in V$ ii) $T(cx) = cT(x)$ $\forall x \in V$.

Thm 1.4. Let V be a vector space with basis $a_1, \cdots a_m$. Let W be a vector space. Given any m vectors $b_1, \cdots b_m \in W$, there is exactly one linear transformation $T: V \to W$ s.t. $\forall i$, $1 \leq i \leq m$, $T(a_i) = b_i$.

* $T: V \to W$

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \text{choose a basis} \updownarrow & & \updownarrow \text{choose a basis} \\ \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^m \end{array}$$

$T(x) = A \cdot x$

Thm 1.5. For any matrix A, the row rank of A equals the column rank of A.

$*$ $\text{rank } T = \dim \operatorname{im} T = \dim \text{col. space}(A) = \dim \text{row. space}(A)$.

Row operation: i) Exchange rows i and j of A $(i \neq j)$

ii) Replace row i of A by itself plus the scalar c times row j. $(i \neq j)$

iii) Multiply row i of A by the non-zero scalar λ.

Thm 1.6. If B is the matrix obtained by applying an elementary row operation to A, then $\text{rank } B = \text{rank } A$. i.e. $B = \begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & 0 & * \cdots & * \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$

$*$ $C = \begin{pmatrix} 1 & 0 & * & 0 & \cdots & * \\ 0 & 1 & * & 0 & \cdots & * \\ 0 & 0 & 0 & 1 & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$

Def. The transposition of matrix A is A^{tr} s.t. if $A = (a_{ij})$ then $A^{tr} = (a_{ji})$

$*$ i) $(A^{tr})^{tr} = A$ ii) $(A+B)^{tr} = A^{tr} + B^{tr}$ iii) $(A \cdot C)^{tr} = C^{tr} \cdot A^{tr}$ iv) $\text{rank } A^{tr} = \text{rank } A$.

$*$ For the following E, EA - row ~~#~~ operation. AE - colm. operation.

$$E = \begin{array}{cc} \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} & \begin{matrix} \\ \\ \text{row } i \\ \\ \text{row } j \\ \\ \\ \end{matrix} \end{array} \quad E' = \begin{array}{cc} \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & \cdots & c & & \\ & & \vdots & & \vdots & & \\ & & 0 & \cdots & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} & \begin{matrix} \\ \\ \text{row } i \\ \\ \text{row } j \\ \\ \\ \end{matrix} \end{array} \quad E'' = \begin{array}{cc} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & & 1 \end{pmatrix} & \begin{matrix} \\ \\ \text{row } i \\ \\ \\ \\ \end{matrix} \end{array}$$

Thm 2.1 Let A be an $n \times m$ matrix. Any elementary row operation on A may be carried out by premultiplying A by the corresponding elementary matrix.

Def. A - $n \times m$ matrix, B & C - $m \times n$ matrices.

B is a $\boxed{\text{left inverse}}$ for A if $B \cdot A = I_m$. C is a $\boxed{\text{right inverse}}$ for A if $A \cdot C = I_n$

Thm 2.2. If A has both a left inverse B and a right inverse C, then they are unique and equal.

Def. A is $\boxed{\text{invertible}}$ if A has both a right inverse and a left inverse.

The unique matrix that is both a right inverse and a left inverse for A is called the $\boxed{\text{inverse}}$ of A and is denoted $\boxed{A^{-1}}$

Thm 2.3. Let A be a matrix of size n by m. If A is invertible, then $n = m = \text{rank } A$.

Thm 2.4. Let A be a matrix of size m by n. $n = m = \text{rank } A \Rightarrow A$ is invertible.

Furthermore: A = a product of elementary matrices.

Thm 2.5. If A is a square matrix, B is left matrix of $A \Rightarrow B$ is also the right inverse.

Def. Let A be an n by n matrix, a function $\det : M_{n\times n} \to \mathbb{R}$ is called a $\boxed{\text{determinant function}}$ if it satisfies following axioms:

i) If B is the matrix obtained by exchanging any two rows of A, then $\det B = -\det A$

ii) Given i, the function $\det A$ is linear as a function of the i^{th} row alone.

iii) $\det I_n = 1$.

$\ast$. $A = (a_{ij})_{1\le i\le n,\, 1\le j\le n}$. $\det A = \sum_{\sigma\in S} \operatorname{sign}\sigma\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$

where S = sets of all bijective functions $\sigma : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$

Thm 2.6. Let A be an $n\times n$ matrix.

a) If E is the elementary matrix corresponding to the operation that exchanges rows i and j then $\det(E\cdot A) = -\det A$.

b) If E' is the elementary matrix corresponding to the operation that replaces row i of A by itself plus c times row j, then $\det(E'A) = \det A$.

c) If E'' is the elementary matrix corresponding to the operation that multiplies row i of A by the non-zero scalar λ, then $\det(E''A) = \lambda \det(A)$.

d) If A is the identity matrix I_n, then $\det A = 1$.

Thm 2.7. Let A be square matrix. If the rows of A are independent, then $\det A \neq 0$. if the rows are dependent, then $\det A = 0$. Thus an $n\times n$ matrix A has rank n iff $\det A \neq 0$.

Thm 2.8. Given a square matrix A, reducing it to echelon form B by elementary row operations of E & E'. If B has a zero row, then $\det A = 0$.
Otherwise, let k be the number of row exchanges involved in reduction process, then $\det A = (-1)^k$ times product of the diagonal entries of B.

Thm 2.9. Let A and B be $n\times n$ matrices. Then $\det(A\cdot B) = (\det A)\cdot(\det B)$.

Thm 2.10. $\det A^{tr} = \det A$

Def. Let A be an $n\times n$ matrix. The $\boxed{(i,j)\text{-minor}}$ of A is a $n-1 \times n-1$ matrix obtained from A by deleting the i^{th} row and the j^{th} column of A, denoted A_{ij}.

Thm 2.11. A is an $n\times n$ matrix of rank n, $B = A^{-1}$, then $b_{ij} = (-1)^{j+i} \det A_{ji} / \det A$

Thm 2.12. Let A be an $n\times n$ matrix, i fixed. then $\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik}\cdot \det A_{ik}$. $\square$

MAT 257 Sep 16.

Topology

Def. Let X be a set. A $\boxed{\text{metric}}$ on X is a function $d: \underbrace{X \times X}_{\text{pairs of points on } X} \to \mathbb{R}$ s.t.

i) $d(x,y) = d(y,x)$ for $x,y \in X$

ii) $d(x,y) \geq 0$ & $d(x,y) = 0$ iff $x = y$

iii) $\forall x, y, z \in X$ $d(x,y) + d(y,z) \geq d(x,z)$ "the triangle inequality"

Example 0 $X = \mathbb{R}$, $d(x,y) = |x-y|$

1 $X = \mathbb{R}^n$, $d_1(x,y) = \|x-y\| = \left(\sum (x_i - y_i)^2\right)^{1/2}$

$d_2(x,y) = |x-y| = \max |x_i - y_i|$

2 X - any set. $d(x,y) = \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases}$

← a vector space it self, dim = ∞

3 $X = C([0,1])$ = set of all continuous functions $f: [0,1] \to \mathbb{R}$

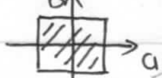
$f, g \in X$ i.e. $f, g: [0,1] \to \mathbb{R}$ define $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$

Note: f, g cont. function $\Rightarrow (f-g)$ cont. $\Rightarrow |f-g|$ cont.; As $x \in [0,1]$ bdd, we can use max.

Def. A metric space is a set X along with a choice of n metric on it.

e.g. $\mathbb{R}$, $\mathbb{R}^n$, $C([0,1])$ ...

Def. Given a metric space X and $x_0 \in X$ and $\varepsilon > 0$; $U(x_0, \varepsilon) = \{x \in X : d(x_0, x) < \varepsilon\}$ is the $\boxed{\varepsilon\text{-neighburhood}}$ (ε-nbd) or the $\boxed{\varepsilon\text{-ball}}$ around x_0.

e.g. $\mathbb{R}^2$, $|\cdot|_{sup}$. $U(0,1) =$  $x = a, b$ $d(x,0) = |x-0| = |x| = \max(|a|, |b|)$.

← a ball looks "square"

Def. A set $U \subset X$ is called $\boxed{\text{open}}$ if $\forall x_0 \in U$, $\exists \varepsilon > 0$ s.t. $U(x_0, \varepsilon) \subset U$

In other words: every pt in U has an ε-nbd contained in U.

A set $F \subset X$ is $\boxed{\text{closed}}$ if $F^c = X \setminus F$ is open.

Note: A set can be both open & closed, or neither closed or open.

← clopen.

Claim: $U(x_0, r)$ is open.

proof: Let $y \in U(x_0, r)$, take $\varepsilon = r - d(x_0, y)$

$\varepsilon > 0$ as $y \in U(x, r)$ then $d(x_0, y) < r$

Claim: $U(y, \varepsilon) \subset U(x_0, r)$

proof: Let $z \in U(y, \varepsilon)$, meaning $d(z, y) < \varepsilon$

then $d(x_0, z) \leq d(x_0, y) + d(y, z) < d(x_0, y) + \varepsilon = d(x_0, y) + r - d(x_0, y) = r$

$\Rightarrow z \in U(x_0, r)$. $\square$ $\square$

Theorem 1a. i) $\emptyset, X$ are open. ← full set.

ii) An arbitrary union of opens is open. $\forall \alpha \in I, U_\alpha$ open $\Rightarrow \bigcup_{\alpha \in I} U_\alpha$ is open. ← can be infinite

iii) $\forall 1 \leq i \leq n$, U_i is open $\Rightarrow \bigcap_{i=1}^{n} U_i$ is open. ← finite sets