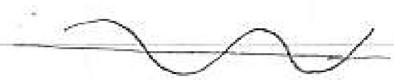
Nov 30, 2012

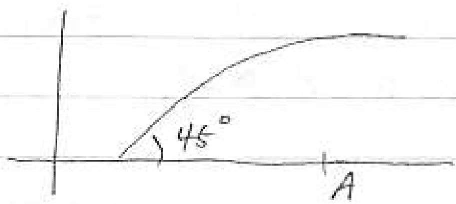
<u>Reminders</u> $y'' + qy = 0$

a "restoring force"

$q < 0$: At most one zero

$q > 0$, $\int_A^\infty q\,dx = \infty$: Oscillations 

$q > 0$



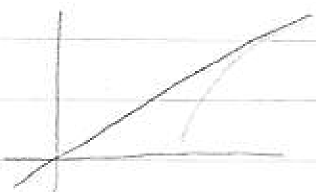
$\int xq\,dx$

<u>Theorem</u>

$A > 0$ q is cont. & positive ($q > 0$) on $[A, \infty)$

$\int_A^\infty x\,q(x)\,dx < \infty$ & $y'' + qy = 0$ Then

1. $\exists B > A$ s.t. y has no zeros past B
2. $\exists$ constant K s.t.



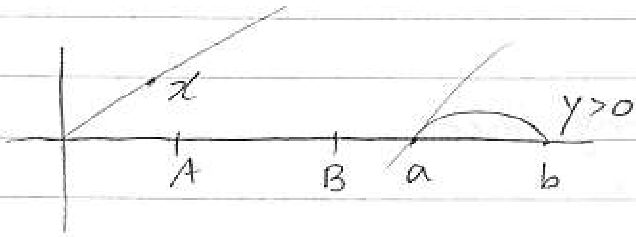
$$\lim_{x \to \infty} y'(x) = K = \lim_{x \to \infty} \frac{y(x)}{x}$$

<u>Comment 1</u> If $K \neq 0$, $(2) \to (1)$

<u>Comment 2</u> Dror could not prove that always $K \neq 0$ and could not find a counterexample

<u>proof (1)</u>

Find $B > A$ s.t. $\int_B^\infty xq\,dx < 1$



assume y has two zeros past B. Call the first a & second b

W.L.O.G $y>0$ on (a,b). also it is frowny & $\alpha = y'(a) > 0$. Also $y(x) \leq \alpha(x-a) \leq \alpha x$

$$\alpha < y'(a) - y'(b) = -\int_a^b y'' \, dx$$

$$= \int_a^b y q \, dx$$

$$\leq \int_a^b \alpha x q \, dx$$

$$\leq \alpha \int_B^\infty x q \, dx$$

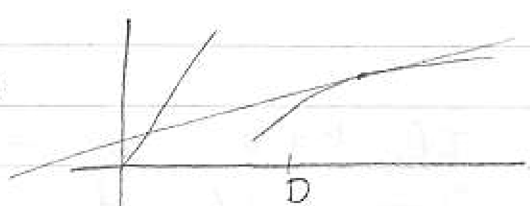
$$< \alpha$$

Contradiction!

<u>proof of (2)</u>

Find $D \geq B$ so $y>0$ on $[D,\infty)$

y is frowny so it lies below one of its tangents so we can find some β s.t $y(x) < \beta x$ on $[D,\infty)$



let $a<b$ be in $[D,\infty)$

$$|y'(a) - y'(b)| = \left| \int_a^b y'' \, dx \right|$$

$$= \int_a^b y q \, dx$$

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$$< \beta \int_a^b x q \, dx$$

$$< \beta \int_a^\infty x q \, dx \xrightarrow[a \to \infty]{} 0$$

So y' is a cauchy function. So
$\exists K$ s.t. $y'(x) \longrightarrow K$
Now by L'Hôpital.

$$\lim_{x \to \infty} \frac{y(x)}{x} = \lim \frac{y'(x)}{1} = K$$

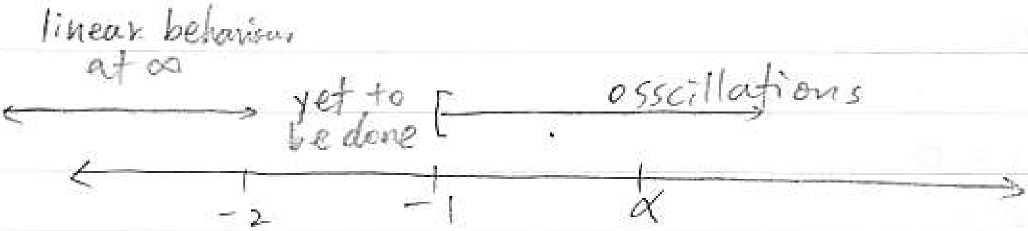
<u>example</u>

Study $\quad y'' + \underbrace{x^\alpha}_{q} y = 0 \quad . \ \alpha \in \mathbb{R} \quad x > 0$

$$\int q \, dx = \int x^\alpha dx = \infty \quad \text{if } \alpha \geq -1$$

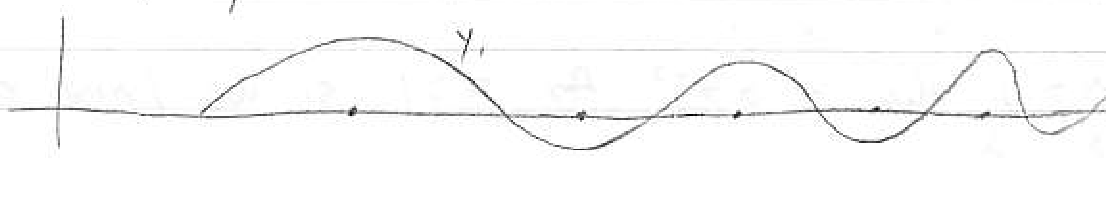
So in that case y oscillates.

$$\int x q(x) dx = \int x^{\alpha+1} dx < \infty \quad \text{if } \alpha+1 < -1 \iff \text{if } \alpha < -2$$



<u>Theorem (Sturm Comparison Theorem)</u>

If $y_1'' + q_1 y_1 = 0$ & $y_2'' + q_2 y_2 = 0$ & $q_2 > q_1$ in some domain. then in the open interval between any two roots of y_1 in our domain, there is a root of y_2



proof

WLOG. $y_1 > 0$ on (a,b) where a & b are two adjacent roots of y_1

$\Rightarrow y_1(a) = y_1(b) = 0$, $y_1'(a) > 0 > y_1'(b)$

By contradiction, assume y_2 has no roots in (a,b)

w.l.o.g $y_2 > 0$ in (a,b)

Consider $W(x) = y_1' y_2 - y_1 y_2'$

$W(a) = y_1'(a) y_2(a) - 0 \geq 0$

$W(b) = y_1'(b) y_2(b) - 0 \leq 0$

yet $W' = y_1'' y_2 + \cancel{y_1' y_2'} - \cancel{y_1' y_2'} - y_1 y_2''$

$= -q_1 y_1 y_2 + q_2 y_1 y_2$

$= (q_2 - q_1) y_1 y_2$

> 0 on (a,b)

So W is increasing

Contradiction!

example

$y'' + x^{-2} y = 0$

$x^2 y'' + y = 0$

Try $y = x^r$

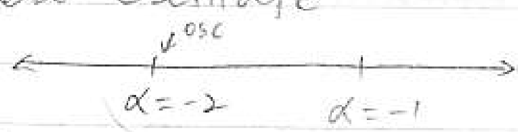
$r(r-1) + 1 = 0$

$r^2 - r + 1 = 0$

$r = \dfrac{1 \pm \sqrt{1-4}}{2}$

solutions : $x^{\frac{1}{2}} \begin{matrix}\cos \\ \sin\end{matrix} \left(\frac{\sqrt{3}}{2} \log x\right)$

these oscillate



But if $\alpha > -2$, then $x^\alpha > x^{-2}$ for $x > 1$, so we have oscillations for all $\alpha \geq -2$