Claim 1

Solutions to $v'(t) = A(t)v(t)$ exist and are unique wherever A is continuous.

Claim 2

If $\psi'(t) = A(t)\psi(t)$ on an interval on which A is cont. then ψ is either regular for all t or singular for all t.

Def + Pf2 of claim 2, using "Wronskian"

proof of claim 1

On $I_0 = [-1000, 1000]$ By compactness

Find μ s.t. $|A_{ij}| \le \mu$ on I_0

Take $t_0 \in I_0$. & some V_0 $(v(t_0) = V_0)$ and let

$R = [t_0 - a, t_0 + a] \times B(V_0, |V_0|)$ where a is biggest that fits & $B(V_0, r) = \{v : N \to V_0 < r\}$. $|A_{ij}| < \mu$ so $v \longmapsto Av$ is uniformly Lips. with $K = n\mu < \infty$. So existence and uniqueness holds on $[t_0 - \delta, t_0 + \delta]$ where $\delta = \min(a, \frac{b}{M})$ in our case. $b = |V_0|$. $M = n\mu 2|V_0|$ so $\frac{b}{M} = \frac{1}{2\mu n} = \varepsilon$

ε is positive and independent of t_0, V_0. So whenever a solution exist and is unique, it exists and is unique for another ε seconds

proof 2 of claim 2

$W(t) = \det \psi(t)$ "The Wronskian"

Aside

Suppose $M(t)$ is a matrix that depends on t, $t \longmapsto M(t)$ is differentiable. $(\det M(t))' = ?$

$$\det(M(t+\varepsilon)) = \det(M(t) + \varepsilon M'(t) + o(t))$$
$$\overset{\text{mod } o(\varepsilon)}{=} \det(M + \varepsilon M')$$
$$= \det(M(I + \varepsilon M^{-1} M'))$$

$$= \det(M)\det(I + \varepsilon M^{-1}M')$$

$$\det\begin{pmatrix} 1+\varepsilon a_{11} & \varepsilon a_{12} & \varepsilon a_{13} & & \\ & 1+\varepsilon a_{22} & \varepsilon a_{23} & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

$$\doteq 1 + \text{contribution coming from the identity} + \varepsilon^2 \cdots$$

$$= (1+\varepsilon a_{11})(1+\varepsilon a_{22})(1+ \cdots)$$

$$= 1 + \varepsilon(a_{11} + a_{22} + a_{33} + \cdots)$$

$$= 1 + \varepsilon\, tr(a_{ij})$$

$$(\det M(t))' = \det(M)\, tr(M^{-1}M')$$

In our case,

$$W(t+\varepsilon) = \det(\psi(t+\varepsilon))$$

$$\overset{\text{mod } \varepsilon^2}{=} \det(\psi + \varepsilon\psi')$$

$$= \det(\psi + \varepsilon A\psi)$$

$$= \det(1+\varepsilon A)\det(\psi)$$

$$= (1+\varepsilon\, trA)\, W$$

$$\rightsquigarrow W' = (trA)W$$

$$W(t) = \left(\exp\int_{t_0}^{t} tr(A(s))\,ds\right) W(t_0)$$

X_0

□

<u>Q</u> what is the corresponding story for
$y'' + p(x)y' + q(x)y = 0$?

$$y'' + py' + qy = 0 \overset{v_1 = y}{\underset{v_2 = y'}{\iff}} \begin{array}{l} v_1' = v_2 \\ v_2' = -pv_2 - qv_1 \end{array}$$

$$\iff v' = \begin{pmatrix} 0 & 1 \\ -q & p \end{pmatrix} v$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

y_1 & y_2 solve the equation

$$W' = -pW$$