Def: $f : X \longrightarrow Y$ is cts at $x_0 \in X$ if for every nbhd V of $f(x_0)$, there is a nbhd B of x_0 s.t. $f(U) \subset V$

an open set V containing $f(x_0)$

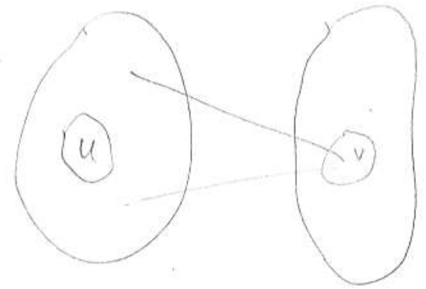
$\Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \quad d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$

Def: f "cts" means "f cts at every x"

Thm: TFAE : (for $f : X \longrightarrow Y$)

1. f cts
2. For every open $V \subset Y$, $f^{-1}(V)$ open
3. For every F closed in Y, $f^{-1}(F)$ closed
4. If $X = Y = \mathbb{R}$, f cts

$f : X \longrightarrow Y$



$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$

$f(U) = \{f(u) \mid u \in U\}$

Claim: $f : X \longrightarrow Y$, $A, B \subset X$, $C, D \subset Y$

1. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
2. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
3. $f^{-1}(D^c) = (f^{-1}(D))^c$
4. $f(A \cup B) = f(A) \cup f(B)$
5. $f(A \cap B) \subset f(A) \cap f(B)$
6. $f(A^c) \supset f(A)^c$

pf : $(1 \Rightarrow 2)$

Let $V \subset Y$ open

Let $x_0 \in f^{-1}(V)$ meaning $f(x_0) \in V$

So V is a nbhd of $f(x_0)$ by cty assumption, there is a nbhd V of x_0 s.t. $f(U) \subset V$. But then

$x_0 \in U \subset f^{-1}(V)$.

So $f^{-1}(V)$ open

$(2 \Rightarrow 1)$

Let $x_0 \in X$. Let V be a nbhd of $f(x_0)$

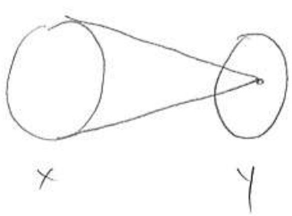
Let $U = f^{-1}(V)$ by assumption U open... clearly $x_0 \in U$

So U a nbhd of x_0 & $f(U) = f(f^{-1}(U)) \subset V$

$(2 \Leftrightarrow 3)$

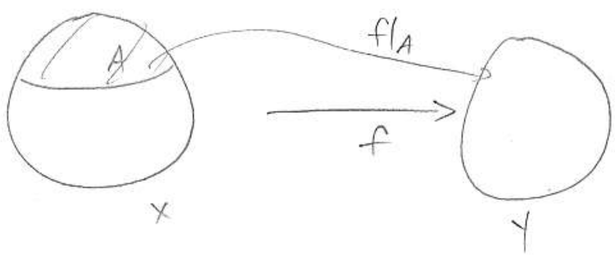
$f^{-1}(F^c) = (f^{-1}(F))^c$

Thm : 1. Constant fct are cts.



2. $I : X \longrightarrow X$ cts

3. $f : X \longrightarrow Y$ cts $\Rightarrow f|_A$ cts



4. $f : X \longrightarrow Y$ cts, $g : Y \longrightarrow Z$ cts $\Rightarrow g \circ f : X \longrightarrow Z$ cts

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

5. $f : X \longrightarrow \mathbb{R}^n$

$f = (f_1, f_2, \cdots, f_n)$

f cts $\Leftrightarrow \forall_i \; f_i$ cts

6. $\pi_i : \mathbb{R}^n \longrightarrow \mathbb{R}$

$\pi_i(x) = x_i$ cts

7. $f, g : X \longrightarrow \mathbb{R}$ cts $\Rightarrow f+g,\ f-g,\ fg,\ \frac{f}{g}$ cts

$A \subset X \longrightarrow \bar{A}$

Def: If $A \subset X$ is a subset, then

$\text{int}\, A = \{ x \in A \mid \exists \varepsilon > 0,\ B(x, \varepsilon) \subset A \}$

= union of all open sets contained in A

= The maximal open set contained in A.

Def: $\text{Ext}(A) = \text{int}(A^c)$

Def: $\underset{\substack{\nearrow \\ \text{boundary}}}{\text{Bd}}\, A = X \setminus (\text{int}\, A \cup \text{ext}\, A)$

<u>Claim</u>: 1. $\text{ext } A = X \setminus \overline{A}$

2. $\text{int } A = X \setminus \overline{A^c}$

3. $\text{Bd } A = \overline{A} \cap \overline{A^c}$