Example

① $y' = \frac{x^2-1}{y^2+1}$ $y(-1)=1$

Want some interval I, $-1 \in I$ and some function $y: I \to \mathbb{R}$ that satisfies the ODE and s.t. $y(-1)=1$

$(y^2+1)y' = x^2-1$

$\Rightarrow \int (y^2+1)(x)\, y'(x)\,dx = \int (x^2-1)\,dx$

$\Rightarrow \frac{y^3(x)}{3} + y(x) = \frac{x^3}{3} - x + c$

$\frac{y^3(-1)}{3} + y(-1) = \frac{(-1)^3}{3} - (-1) + c$

$\frac{1}{3} + 1 = -\frac{1}{3} + 1 + c \Rightarrow c = \frac{2}{3}$

$\frac{y^3(x)}{3} + y(x) = \frac{x^3}{3} - x + \frac{2}{3}$

② $0 = (\cos(2y) - \sin x)\,dx - 2\tan x \sin(2y)\,dy$

$M dx + N dy = 0$

$M + N y' = 0$

$M(x,y) = \cos(2y) - \sin x$, $N = -2\tan x \sin(2y)$

$M_y = -2\sin 2y$, $N_x = -2\sec^2 x \sin(2y)$

$(\mu M)_y = (\mu N)_x$

$\mu_y M + \mu M_y = \mu_x N + \mu N_x$

$\mu(M_y - N_x) = \mu_x N - \mu_y M$

Say $\mu_x = 0$

$\frac{M_y - N_x}{M} = -\frac{\mu_y}{\mu}$

$\frac{M_y - N_x}{M} = \frac{-2\sin(2y) + 2\sec^2 x \sin(2y)}{\cos 2y - \sin x}$

$$\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}$$

$$= \frac{-2\sin(2y) + 2\sec^2 x \sin(2y)}{-2\tan x \sin(2y)}$$

$$= \frac{-2 + 2\sec^2 x}{-2\tan x}$$

$$= \frac{1-\sec^2 x}{\tan x}$$

$$\boxed{1-\sec^2 x = 1 - \frac{1}{\cos^2 x} = \frac{\cos^2 x - 1}{\cos^2 x} = -\frac{\sin^2 x}{\cos^2 x} = -\tan^2 x}$$

$$= \frac{-\tan^2 x}{\tan x}$$

$$= -\tan x$$

$\log|\mu| = \log|\cos x|$

$|\mu| = |\cos x|$

$\mu = \cos x$

want F s.t. $F_x = \mu M$, $F_y = \mu N$

$$F = \int \mu M\,dx = \int (\cos x \cos(2y) - \cos x \sin x)\,dx$$

$$= \int \left(\cos x \cos(2y) - \frac{\sin 2x}{2}\right) dx$$

$$= \sin x \cos(2y) + \frac{\cos(2x)}{4} + g(y)$$

$F_y = -2\sin x \sin(2y) + g'(y)$

$= -2\cos x \tan x \sin(2y)$

$\Rightarrow -2\sin x \sin(2y) + g'(y) = -2\sin x \sin(2y)$

$\Rightarrow g'(y) = 0$

$\Rightarrow g(y) = c \ \forall y$

$$F(x,y) = \sin x \cos(2y) + \frac{\cos(2x)}{4} + C_1$$

$F(x,y) = C$

y solves ODE iff $\sin x \cos(2y) + \dfrac{\cos(2x)}{4} = C_2 \ \forall x$

$\sin x \cos(2y) = C_2 - \frac{\cos(2x)}{4}$

$\cos(2y) = \frac{C_2}{\sin x} - \frac{\cos(2x)}{4\sin x}$

$y = \frac{1}{2}\arccos\left(\frac{C_2}{\sin x} - \frac{\cos(2x)}{4\sin x}\right)$

③ $3x + \frac{6}{y} + \left(\frac{x^2}{y} + \frac{3y}{x}\right)y' = 0$

If $\frac{N_x - M_y}{xM - yN}$ depends on xy then $Mdx + Ndy = 0$ has an integrating factor $\mu(x,y) = \lambda(xy)$

$M = 3x + \frac{6}{y}$, $N = \frac{x^2}{y} + \frac{3y}{x}$

$M_y = -\frac{6}{y^2}$. $N_x = \frac{2x}{y} - \frac{3y}{x^2}$

$$\frac{N_x - M_y}{xM - yN} = \frac{\frac{2x}{y} - \frac{3y}{x^2} + \frac{6}{y^2}}{3x^2 + \frac{6x}{y} - x^2 - \frac{3y^2}{x}}$$

$$= \frac{2x^3y - 3y^3 + 6x^2}{3x^4y^2 + 6x^3y - x^4y^2 - 3xy^4}$$

$$= \frac{2x^3y - 3y^3 + 6x^2}{2x^4y^2 + 6x^3y - 3xy^4}$$

$$= \frac{1}{xy}\frac{2x^3y - 3y^3 + 6x^2}{2x^3y + 6x^2 - 3y^3}$$

So $\exists$ integrating factor $\mu(x,y) = \lambda(xy)$

$(\mu M)_y = (\mu N)_x$

$x\lambda'(xy)M + \lambda(xy)M_y = y\lambda'(xy)N + \lambda(xy)N_x$

$\lambda'(xy)(xM - yN) = \lambda(xy)(N_x - M_y)$

$$\frac{\lambda'(xy)}{\lambda(xy)} = \frac{N_x - M_y}{xM - yN} = \frac{1}{xy}$$

$z = xy$

$$\frac{\lambda'(z)}{\lambda(z)} = \frac{1}{z}$$

$\log|\lambda(z)| = \log|z|$

$\lambda(z) = z$

xy