Def. $\phi \in \mathcal{L}^k(V)$ is "alternating" if it switches sign whenever you swap two of its arguments
$$\phi(\dots x, \dots y \dots) = -\phi(\dots y \dots x \dots)$$

Claim: ϕ is alt. iff $\phi(\dots x \dots x \dots) = 0$

$\phi = \mathcal{L}^2 \;:\; \Rightarrow$

$$\phi(x,x) = -\phi(x,x) \quad \Rightarrow \quad \phi(x,x) = 0.$$

Feb 06

Let V be a vector space with basis $a_1 \cdots a_n$

Thm. $\forall I \in \underline{n}^k . \; \exists! \, \phi_I \in \mathcal{L}^k(V)$ s.t. $\forall J \in \underline{n}^k \;\; \phi_I(a_J) = \delta_{IJ}$; $\{\phi_I\}$ is a basis of $\mathcal{L}^k(V)$

Def. $\psi \in \mathcal{L}^k(V)$ is "alternating" means $\psi(\dots x \dots y \dots) = -\psi(\dots y \dots x \dots)$

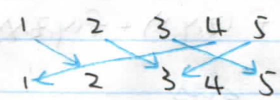
claim: ψ is alternating iff $\psi(\dots x \dots x \dots) \overset{\leftarrow \text{always}}{\equiv} 0$

pf: $(\Rightarrow)$ done

$$(\Leftarrow)\; 0 = \psi(\dots x+y \dots x+y \dots) = \underbrace{\psi(\dots x \dots x \dots)}_{=0} + \psi(\dots x \dots y \dots) + \psi(\dots y \dots x \dots) + \underbrace{\psi(\dots y \dots y \dots)}_{=0}$$
$$= \psi(\dots x \dots y \dots) + \psi(\dots y \dots x)$$
$$\Rightarrow \psi(\dots x \dots y \dots) = -\psi(\dots y \dots x \dots) \qquad \square.$$

Def. S_n = "the permutation group of $\underline{n}$" $= \{\sigma : \underline{n} \circlearrowleft ; \sigma \text{ is 1-1 \& onto}\}$

e.g. $S_5 \ni \sigma$:



$|S_n| = n! = 1 \cdot \dots \cdot n$

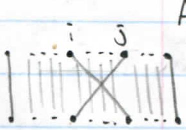
S_n is the "Symmetric group of order n" : $\sigma, \tau \in S_n$, $\sigma \cdot \tau = \sigma \circ \tau \quad (\sigma\tau)\, i = \sigma(\tau(i)) = \sigma\tau i$

1) Assoc: not communitative 2) $\exists$ identity. 3) Every perm has inverse.

Def. $A^k(V) := \{\psi \in \mathcal{L}^k : \psi \text{ is alternating}\}$

Claim: If $\psi \in \mathcal{L}^k(V)$ then $\psi \in A^k(V) \Leftrightarrow \forall \sigma \in S_k \;\; \psi^\sigma(v_1 \dots v_k) := \psi(v_{\sigma 1}, \dots, v_{\sigma k})$
$$= (-1)^\sigma \cdot \psi(v_1 \dots v_k).$$

Aside: There is a unique sign assignment. $\text{sign}: S_n \to \{\pm 1\}$

(ij) 

(Usually write $\text{sign}(\sigma) = (-1)^\sigma = (-1)^\sigma$)

Such that 1. $(-1)^{(ij)} = -1$ $\quad (ij) \in S_n$ as $(ij)(k) = \begin{cases} j & k=i \\ i & k=j \\ k & \text{otherwises.} \end{cases}$ i.e. map $i \mapsto j$ & $j \mapsto i$

2. $(-1)^{\sigma\tau} = (-1)^\sigma (-1)^\tau$

$$(-1)^\sigma := \prod_{1 \le i < j \le n} \text{sign}(\sigma_j - \sigma_i)$$

NTS (need to show) $(-1)^{(ij)} = -1 \qquad (-1)^{\sigma\tau} = (-1)^\sigma (-1)^\tau \leftarrow$ additive $\Rightarrow$ composition additive.

$\sigma \in S_n \to A_\sigma \in M_{n \times n}. \qquad A_\sigma = \left| e_{\sigma_1} \middle| e_{\sigma_2} \middle| \cdots \middle| e_{\sigma_n} \right| \qquad n=2 \quad A_{(12)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$(-1)^\sigma = \det(A_\sigma)$$

pf. (claim above) Obvious.

$\mathcal{L}^k \rightsquigarrow A^k \quad \underline{n}^k \xrightarrow{\text{ascending}} \underline{n}^k_a \qquad \underline{n}^k_a = \{(i_1, \dots i_k) \in \underline{n}^k ; i_1 < i_2 < \dots < i_k\}$

$|\underline{n}^k_a| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Thm: $\forall I \in \binom{n}{k} \ \exists ! \psi_I \in A^k(V)$ s.t. $\forall J \in \binom{n}{k}$. $\psi_I(a_J) = \delta_{IJ}$;

$\{\psi_I\}_{I \in \binom{n}{k}}$ is a basis of $A^k(V)$ $\dim(A^k) = \binom{n}{k}$

Feb. 08

$S_n = \{\text{bijections } \sigma : \underline{n} \circlearrowright\}$, $\exists$ unique sign: $S_n \to \{\pm 1\}$ s.t. $(-1)^{\sigma \circ \tau} = (-1)^\sigma (-1)^\sigma$, $(-1)^{(ij)} = -1$.

Thm. $\forall I \in \binom{n}{k}$ $\exists ! \psi_I \in A^k(V)$ s.t. $\psi_I(a_J) = \delta_{IJ}$. These $\{\psi_I\}$ make a basis of A^k; hence $\dim A^k(V) = \binom{n}{k}$

$f \in A^k(V)$ $f^\sigma(x_1, \dots x_k) = f(x_{\sigma_1}, \dots x_{\sigma k})$. $f^\sigma = (-1)^\sigma f$

$\binom{n}{k} = (i_1, i_2, \dots, i_k)_{1 \le i_\alpha \le i_{\alpha+1} \le n}$. $\binom{X}{k}$ = subsets of ~~size~~ side k of X.

Def. $\psi_I = \sum_{\sigma \in S_k} (-1)^\sigma \phi_I^\sigma \in \mathcal{L}^k(V)$.

Claim: ψ_I is in $A^k(V)$. $(\psi_I)^\tau = (\sum_{\sigma \in S_k} (-1)^\sigma \phi_I{}^\sigma)^\tau = \sum_{\sigma \in S_k} (-1)^\sigma (\phi_I^\sigma)^\tau = \sum_{\sigma \in S_k} (-1)^\sigma \phi_I^{\tau \cdot \sigma}$ $\quad \pi = \tau \sigma \quad \tau^{-1}\pi = \sigma$

$= \sum_{\pi \in S_k} (-1)^{\tau^{-1}\pi} \phi_I^\pi$

Aside: $f, g \in \mathcal{L}^k$, $\sigma \in S_k$. $(af + bg)^\sigma = af^\sigma + bg^\sigma$

Aside: $((f)^\sigma)^\tau = f^{\tau \cdot \sigma}$

$(-1)^{\tau^{-1}} \cdot (-1)^\pi = (-1)^\tau (-1)^\pi$

$= (-1)^\tau \sum_{\pi \in S_k} (-1)^\pi \phi_I^\pi = (-1)^\tau \psi_I$.

$f(x,y) \rightsquigarrow f(x,y) - f(y,x) = g(x,y) \qquad g(y,x) = f(y,x) - f(x,y) = -g(x,y)$

$f(x,y,z) \rightsquigarrow f(x,y,z) - f(y,x,z) - f(x,z,y) - f(z,y,x) + f(y,z,x) + f(z,x,y)$.

~~Def. $\psi_I = \sum_{\sigma \in S_k} (-1)^\sigma \phi_I^\sigma \in \mathcal{L}^k$~~

$\psi_I(a_J) = \sum_{\sigma \in S_k} (-1)^\sigma \phi_I^\sigma (a_{j_1}, \dots a_{j_k}) = \phi_I(a_J) + \underbrace{\sum \pm \phi_I \text{ ~~(list of variables~~ not in ascending order)}}_{=\delta_{IJ}}$.

$I, J \in \binom{n}{k}$ $I = (i_1 < \dots < i_k)$. $J = (j_1 < \dots < j_k)$. (In general $\phi_I(a_K) = \delta_{IJ}$

If ψ_I is alternating, it is determined by its values on ascending sequences of basis vectors.

$\Rightarrow \psi_I(a_J)$ $J \in \binom{n}{k}$. alternating ψ_I

Claim: $\{\psi_I\}$ is lin. indep. in $A^k(V)$.

Pf. Assume $\sum_{I \in \binom{n}{k}} \alpha_I \psi_I = 0$. where $\alpha_I \in \mathbb{R}$ Evaluate both sides on a_J, where $J \in \binom{n}{k}$.

$\sum_I \alpha_I \psi_I(a_J) = \alpha_J = 0$ □.

Claim: $\{\psi_I\}$ spans $A^k(V)$.

Pf. Take $f \in A^k(V)$. $[f = \sum \alpha_I \psi_I$ and on a_J. $f(a_J) = \alpha_J]$.

s.t. $\alpha_J = f(a_J)$, and then $f = \sum \alpha_J \psi_J$ Indeed, it's enough to verify this on $a_I, I \in \binom{n}{k}$

$f(a_I) = \sum_J \alpha_J \psi_J(a_I) = \alpha_I$. □.

Example. $V = \mathbb{R}^3$ $(a_1, a_2, a_3) = (e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdots e_3)$

$\phi_i(a_j) = \delta_{ij}$ $\quad \phi_1\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \phi_1(x_1e_1 + x_2e_2 + x_3e_3) = x_1$. $\phi_2\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_2$. $\phi_3\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_3$.

$I = (1\ 2)$ $n=3, k=2$. $\phi_I\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\right) = x_1y_2 \qquad \phi_{(2,1)}(\vec{x}, \vec{y}) = x_2y_1$

$\psi_{(1\,2)}(\vec{x}, \vec{y}) = \phi_{(1\,2)}(\vec{x}, \vec{y}) - \phi_{(2\,1)}(\vec{x}, \vec{y}) = x_1y_2 - x_2y_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

Next: $\psi_I(\dots) = \det(\quad)_K$.

Feb 10.

Thm. $\forall I \in \binom{n}{k}$ $\exists!\, \psi_I \in A^k(V)$ s.t. $\forall J \in \binom{n}{k}$ $\psi_I(a_J) = \delta_{IJ}$ $\{\psi_I\}$ make a basis hence $\dim A^k(V) = \binom{n}{k}$

$*\ \mathbb{R}^3$ $I = (1,2)$ $\psi_I\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = x_1 y_2 - x_2 y_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$. general.

Claim: In $V = \mathbb{R}^n$ with $a_i = e_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$: given $I \in \binom{n}{k}$.

$$\psi_I(x_1 \dots x_k) = \det(X_I) = \det \begin{pmatrix} x_{1,i_1} & x_{2,i_1} & \cdots & x_{k,i_1} \\ x_{1,i_2} & & & x_{k,i_2} \\ \vdots & & & \vdots \\ x_{1,i_k} & \cdots & \cdots & x_{k,i_k} \end{pmatrix}$$

where X_I indicates the matrix formed from rows $(i_1, i_2, \dots, i_k)$ of the matrix $X = (x_1 | \cdots | x_k)$.

Pf: Both sides are alternating & multi-linear in $x_1 \dots x_k$, so both sides are determined by their values on a_J, $J \in \binom{n}{k}$ $\quad \text{lhs}(a_J) = \psi_I(a_J) = \delta_{IJ}$.

$\text{rhs}(a_J) = \det(e_{j_1} | \dots | e_{j_k})_I$.

$= \det \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{cases} \det(I_{k\times k}) = 1 & \text{if } I = J \\ \det\begin{pmatrix}\text{some 0-1 matrix} \\ \text{with a row of 0}\end{pmatrix} = 0 & \text{if } I \neq J \end{cases}$ $\square$

$\mathbb{R}^3$ $\quad A^0 = \text{span}(\psi_{()})$ $\quad \psi_{()}() = 1 \quad 1$

$A^1 = \text{span}(\psi_{(1)}, \psi_{(2)}, \psi_{(3)})$ $\quad \psi_i\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_i \quad 3$

$A^2 = \text{span}(\psi_{12}, \psi_{13}, \psi_{23})$ $\quad \psi_{13}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} \quad 3.$

$A^3 = \text{span}(\psi_{123})$ $\quad \psi_{23}(-) = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$

$A^4 = \{0\}$. $\quad \psi_{123}(x, y, z) = |x||y||z|. \quad 1$

$\otimes$ $\underset{f}{\mathcal{L}^k(V)} \times \underset{g}{\mathcal{L}^l(V)} \longrightarrow \underset{f\otimes g}{\mathcal{L}^{k+l}(V)}$ $\quad (f \otimes g)(x_1 \dots x_{k+l}) = f(x_1 \cdots x_k)\, g(x_{k+1} \cdots x_{k+l})$

Suppose f, g are in A^k, A^l resp.

Is $f \otimes g \in A^{k+l}$? No. $\quad \underset{A^{k+l}}{(\underset{A^1}{\psi_1} \otimes \underset{A^1}{\psi_2})}(x, y) = x_1 y_2$.

Def. If $f \in A^k$, $g \in A^l$ $\quad (f \wedge g)(x_1 \cdots x_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (f \otimes g)^\sigma (-1)^\sigma$

$$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} f(x_{\sigma_1} \cdots x_{\sigma_k}) \cdot (-1)^\sigma \cdot g(x_{\sigma_{k+1}} \cdots x_{\sigma_{(k+l)}})$$

$$= \sum_{\sigma \in S_{k+l}} (-1)^\sigma f(\cdots)\, g(\cdots) \quad \sigma_1 < \sigma_2 < \cdots < \sigma_k,\ \sigma_{k+1} < \cdots < \sigma_{k+l}.$$

In $A^*(\mathbb{R}^4)$ $\quad (\underset{A^2}{\psi_{(13)}} \wedge \underset{A^1}{\psi_4})(x_1, x_2, x_3) = (\psi_{13})(x_1, x_2) \cdot \psi_4(x_3)$ $\quad$ ($\in A^3$)

$\sigma \in S_4$ have $\sigma_1 < \sigma_2$, $\sigma_3 \dots$ $\quad - (\psi_{13})(x_1, x_3)\, \psi_4(x_2) + (\psi_{13})(x_2, x_3)\, \psi_4(x_1)$

Feb 13.

Read: Sec 27-29

Today: Wedge products, target vectors. $f \in A^k(V)$, $g \in A^l(V)$, define $f \wedge g \in A^{k+l}(V)$ via

$$(f \wedge g)(x_1 \dots x_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma f(x_{\sigma_1} \dots x_{\sigma_k})\, g(x_{\sigma(k+1)} \cdots x_{\sigma(k+l)})$$

$$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma_1 < \dots < \sigma_k \\ \sigma_{k+1} < \dots < \sigma_{k+l}}} (-1)^\sigma f(x_{\sigma_1} \cdots x_{\sigma_k})\, g(x_{\sigma_{k+1}} \cdots x_{\sigma_{k+l}}).$$

$f \wedge g(x_1, x_2, \dots x_5) = \frac{1}{3!\,2!} \sum_{\sigma \in S_5} \binom{1\ 2\ 0}{\text{terms}}$ $\quad f \in A^3(V)$ $\quad g \in A^2(V)$. $\quad \left|\left\{ \begin{matrix} 1 & 2 & 3\ 4\ 5 \\ 1 & 2 & 3\ 4\ 5 \end{matrix} \right\}\right| = \binom{5}{3} = 10$

$$= \sum_{\substack{\sigma \in S_5 \\ \sigma_1 < \sigma_2 < \sigma_3 \\ \sigma_4 < \sigma_5}} \binom{10}{\text{terms}} = \dots + (-1)^\sigma f(x_2, x_4, x_5) \cdot g(x_1, x_3) + \dots$$