UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

DECEMBER EXAMINATIONS 2009

Math 240H1 Algebra I — Final Exam

Dror Bar-Natan

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Solve all of the following 5 questions. The questions carry equal weight though different parts of the same question may be weighted differently.

Duration. You have 3 hours to write this exam.

Allowed Material. Basic calculators, not capable of displaying text or sounding speech.

Note. This edition of the exam was slightly modified to reflect minor issues discovered in grading.

Good Luck!

Problem 1. Let V be some vector space over a field F and let W_1 and W_2 be subspaces of V.

1. Prove that $W_1 \cap W_2$ is also a subspace of V.

2. Denote the set of all sums $w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$ by $W_1 + W_2$. Prove that $W_1 + W_2$ is a subspace of V.

3. If in addition it is known that both W_1 and W_2 are finite dimensional, prove that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

   Hint. You may want to start with a basis β of $W_1 \cap W_2$ and extend it to a basis β_1 of W_1 and a basis β_2 of W_2.

Tip. Quote any theorem you use!

Problem 2. State and prove the "dimension theorem", also known as the "rank-nullity theorem", for a given linear transformation $T : V \to W$.
Tip. As always in math exams, when proving a theorem you may freely assume anything that preceded it but you may not assume anything that followed it.

Problem 3.

1. Let $L : P_3(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation given by $L(p) = \begin{pmatrix} p(-2) \\ p(1) \\ p(3) \end{pmatrix}$. Find the matrix A representing L relative to the basis $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$ and the standard basis of $\mathbb{R}^3$.

2. Let $w = a + bi$ be a complex number and let $T : \mathbb{C} \to \mathbb{C}$ be defined by $T(z) = w \cdot z$. Considering $\mathbb{C}$ as a vector space over $\mathbb{R}$, find the matrix B representing T relative to the basis $\{1, i\}$ of $\mathbb{C}$.

Tip. Neatness, cleanliness and organization count!

Problem 4. Find all the solutions (if any exist) of the following two systems of linear equations:

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 1 & -2 & 0 & -1 \\ 1 & -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & 1 & 3 \\ 1 & -2 & 0 & -1 \\ 1 & -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Tip. Show all intermediate steps!

Problem 5. In a very clean, neat and tidy table on your solutions notebook, indicate for each of the following statements if it is true or false.

1. The function $\det : M_{n\times n}(F) \to F$ is a linear transformation.
2. The determinant of a square matrix can be evaluated using an expansion along any row or column.
3. If two rows of a square matrix A are identical then $\det(A) = 0$.
4. If B is a matrix obtained from a square matrix A by interchanging any two columns, then $\det(B) = -\det(A)$.
5. If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $\det(B) = \det(A)$.
6. If B is a matrix obtained from a square matrix A by adding c times one row to another row, then $\det(B) = c\det(A)$.
7. If $A \in M_{n\times n}(F)$ has rank n, then $\det A \neq 0$.
8. The determinant of a square upper triangular matrix (a square matrix A for which $A_{ij} = 0$ whenever $i > j$) is equal to the product of its diagonal entries.
9. For any square matrix A, $\det(A) = \det(A^T)$.
10. For any invertible square matrix A, $\det(A) = \det(A^{-1})$.

Tips. No need to copy the statements to your notebook, and no need to justify your answers. Note that a statement is considered "true" only if it is **always** true, and not just **sometimes**.
Note. In grading I realized that there should have been a penalty for wrong answers, and it should have been allowed not to answer some of the parts, to prevent wild guessing.

Good Luck!