Theorem 1a. i) $\emptyset$, X ← full set. are open.

ii) An arbitrary union of opens is open. $\forall \alpha \in I$, U_α open $\Rightarrow \bigcup_{\alpha \in I} U_\alpha$ is open. ← can be infinite

iii) $\forall 1 \le i \le n$, U_i is open $\Rightarrow \bigcap_{i=1}^{n} U_i$ is open. ← finite many sets.

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proof: (of thm 1a) i) already done.

ii) Let x be some pt. in $\bigcup_{\alpha \in I} U_\alpha$. Meaning $\exists \alpha_0 \in I$ s.t. $x \in U_{\alpha_0}$
But then let $\varepsilon > 0$ be s.t. $U(x, \varepsilon) \subset U_{\alpha_0} \subset U_{\alpha \in I} U_\alpha$ $\square$.

iii) Let $x \in \bigcap_{i=1}^{n} U_i \Rightarrow \forall i, x \in U_i$, for each i choose ε_i ~~s.t.~~.
s.t. $U(x, \varepsilon_i) \subset U_i$.
Let $\varepsilon = \min_{1 \le i \le n} \varepsilon_i$; $\varepsilon > 0$ and $\forall i$, $U(x,\varepsilon) \subset U(x, \varepsilon_i) \subset U_i$
$\Rightarrow U(x, \varepsilon) \subset \bigcap U_i$ $\square$

Theorem 1b. i) $\emptyset$, X are closed.

ii) $\forall \alpha$. F_α is closed $\Rightarrow \bigcap_\alpha F_\alpha$ is closed.

iii) $\forall 1 \le i \le n$, F_i closed $\Rightarrow \bigcup_{i=1}^{n} F_i$ is closed

Theorem 4a,b. $\mathbb{R}^n$, $d_1(x,y) = \|x-y\|$, $d_2(x,y) = |x-y|$

a. U is open relative to $d_1 \Leftrightarrow U$ is open rel. d_2.

b. F is closed rel d_1, $\Leftrightarrow U$ is open rel. d_2.

proof (a) Assume U is d_1-open. Let $x \in U \Rightarrow \exists \varepsilon$ s.t. $U_{d_1}(x, \varepsilon) \subset U$ $\quad (d_2 \le d_1 \le \sqrt{n}\, d_2)$

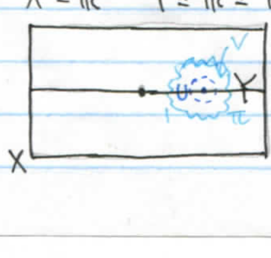
Claim: $U_{d_2}(x, \varepsilon) \supset U_{d_1}(x, \varepsilon) \supset U_{d_2}(x, \varepsilon/\sqrt{n})$

proof: if $y \in U_{d_1}(x, \varepsilon)$ then $d_2(x,y) \le d_1(x,y) < \varepsilon$, so $y \in U_{d_2}(x, \varepsilon)$.
if $y \in U_{d_2}(x, \varepsilon/\sqrt{n})$ then $d_2(x,y) < \varepsilon/\sqrt{n}$
$d_1(x,y) \le \sqrt{n}\, d_2(x,y) < \varepsilon \Rightarrow y \in U_{d_1}(x, \varepsilon)$
Now: $U_{d_2}(x, \varepsilon/\sqrt{n}) \subset U_{d_1}(x, \varepsilon) \subset U \Rightarrow U$ is d_2-open
Need to show: d_2-open $\Rightarrow d_1$-open, d_1-closed $\Rightarrow d_2$-closed $\square$

Theorem 2. X metric space, $Y \subset X$ is also a metric space

1. $U \subset Y$ is open iff $\exists V \subset X$ s.t. V is open in X & $U = V \cap Y$
2. $F \subset Y$ is closed iff $\exists G \subset X$ s.t. G is closed in X & $G \cap Y = F$ $\square$.

Example $X \equiv \mathbb{R}^2$ $\quad Y = \mathbb{R} = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$



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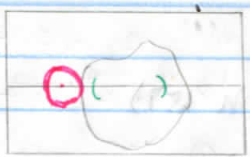
Example 

$\mathbb{R}$ is closed in $\mathbb{R}^2$ but not open.
B is open in A not in X
B is not closed in A nor in X.

$\mathbb{R}^2 = X \supset \mathbb{R} = A \supset (-1,1) = B$

Example 2 $x_1 \overset{n}{\cdots\cdots} x_2$

open sets: $\emptyset$, $\{x_1, x_2\}$, $\{x_1\}$, $\{x_2\}$.
closed sets: SML

Example 3. 

Def. $x_0 \in X$ is called a $\boxed{\text{limit point}}$ of $A \subseteq X$ if $\forall \varepsilon > 0 : (U(x_0, \varepsilon) \cap A) \setminus \{x_0\} \neq \emptyset$.
$\Leftrightarrow$ $\forall \varepsilon > 0$, $|U(x_0, \varepsilon) \cap A| = \infty$ (proof easy)

Def. The closure $\bar{A}$ of a set A. $\bar{A} := A \cup \{\text{limit pts of } A\} = A \cup lp(A)$

Examples 1. In $\mathbb{R}$ $\overline{(0,1)} = [0,1]$
2. $lp\{\frac{1}{n}\} = \{0\}$.

Thm. A is closed $\Leftrightarrow$ $A = \bar{A}$

proof: Same to show A^c open $\Leftrightarrow$ $lp A \subset A$ As $lp A \subset A \Leftrightarrow \bar{A} = A$
($\Rightarrow$) by contradiction suppose $x_0 \in lpA \cap A^c$
Find $\varepsilon > 0$ s.t. $U(x_0, \varepsilon) \subset A^c$ but then $U(x_0, \varepsilon) \cap A = \emptyset$, contradicting $x_0 \in lpA$
($\Leftarrow$) Exercises.

Exercises. $\bar{A}$ is
i) The smallest closed set containing $\bar{A}$.
ii) The intersection of all closed sets containing A.

Def.
Let $F: X \to Y$ $((X, d_X), (Y, d_Y))$. Let $x_0 \in X$ we say that F is cont. at x_0 if there exist a nbd U of x_0 s.t. $F(U) \subset V$.
$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_X(x_0, x) < \delta \Rightarrow d_Y(F(x_0), F(x)) < \varepsilon$
$\Leftrightarrow$ If $X = Y = \mathbb{R}$ cont. is in 157

Example. $X = C([0,1])$ $Y = \mathbb{R}$ $\phi: C([0,1]) \to \mathbb{R} = Y$ by $\phi(F) = F(0.3)$
Claim ϕ is cont. when $F \in C([0,1])$.

Thm. TFAE. for $f: X \to Y$ i) F is cont. (at every pt of X).

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Thm: TFAE. (for $f: X \to Y$).

i) F is cont. (If $X = Y = \mathbb{R}$, f is cont. as in 157)

ii) For every open $V \subset Y$, $f^{-1}(V)$ is open.

iii) For every F closed in Y, $f^{-1}(F)$ is closed.

* $F: X \to Y$. $f^{-1}(V) = \{x \in X : f(x) \in V\}$

$f(U) = \{f(u) : u \in U\}$

Claim: $f: X \to Y$, $A, B \subset X$, $C, D \subset Y$.

i) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

ii) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

iii) $f^{-1}(D^c) = (f^{-1}(D))^c$

iv) $f(A \cup B) = f(A) \cup f(B)$.

v) $f(A \cap B) \subset f(A) \cap f(B)$ $\qquad \phi \neq [\,]$

vi) $f(A^c) \supset f(A)^c$

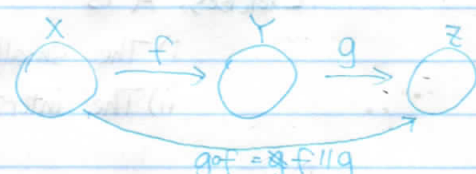
proof (of theorem) i) ⇒ ii): Let $V \subset Y$ be open, let $x_0 \in f^{-1}(V)$, meaning $f(x_0) \in V$, so V is a nbd of $f(x_0)$, by cont. assumption, there is a nbd U of x_0 s.t. $f(U) \subset V$. But then $x_0 \in U \subset f^{-1}(V)$. So $f^{-1}(V)$ is open.

ii) ⇒ i) let $x_0 \in X$, let V be a nbd of $f(x_0)$. Let $U = f^{-1}(V)$, by assumption U is open, clearly $x_0 \in U$ so U is a nbd of x_0 & $f(U) = f(f^{-1}(V)) \subset V$ as required.

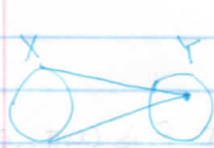
ii) ⇒ iii) $f^{-1}(F^c) = (f^{-1}(F))^c$

iii) ⇒ ii) Same. □.

Thm. (technical) i) Constant functions are cont.



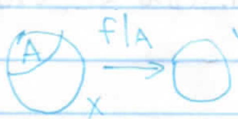
ii) $I: X \to X$ is cont.



iii) $f: X \to Y$ cont. ⇒ $f|_A$ is cont.

iv) $f: X \to Y$ $g: Y \to Z$ ⇒ $f \circ g = f // g$ is cont. (both cont.)

v) $F: X \to \mathbb{R}^n$ $f = (f_1, f_2, f_3, \dots f_n)$ $\quad$ f cont. ⇔ $\forall i$ F_i is cont.



vi) $\pi_i : \mathbb{R}^n \to \mathbb{R}$ $\pi_i(x) = x_i$

vii) $f, g: X \to \mathbb{R}$ cont. ⇒ $f+g$, $f \cdot g$, $f-g$, f/g are cont.

$A \subset X \Rightarrow \bar{A}$

Def. If $A \subset X$ is a subset, $\operatorname{int} A = \{x \in A;\ \exists\, \varepsilon > 0,\ U(x, \varepsilon) \subset A\}$

$\overset{!}{=}$ Union of all open sets ~~containing~~ contained in A $\overset{!}{=}$ The Maximal open set contained in A

Def. $\mathrm{Ext}(A) = \mathrm{int}(A^c)$

Def. $\mathrm{Bd}\,A = X \setminus (\mathrm{int}(A) \cup \mathrm{ext}(A))$ ← boundary

claim: i) $\mathrm{ext}(A) = X \setminus \bar{A}$

ii) $\mathrm{int}\,A = X \setminus \overline{A^c}$

iii) $\mathrm{Bd}\,A = \bar{A} \cap \overline{A^c}$