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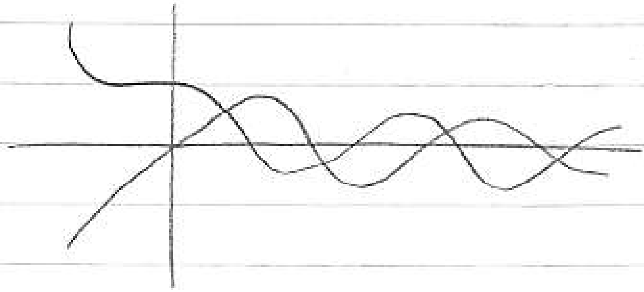
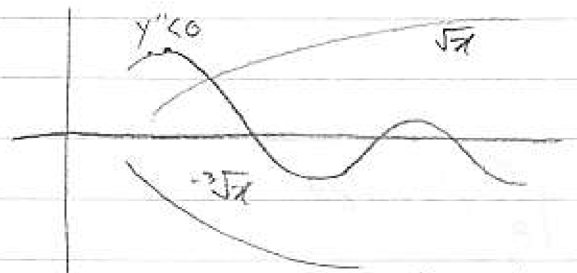
Airy's equation $y'' + xy = 0$



why?

Start by looking at $y'' + qy = 0$ $q = q(x)$

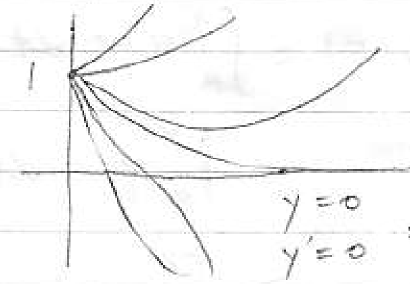
$q > 0$ $y'' = -qy$



q is a "restoring force"

$q < 0$ "inflationary force"

if $y > 0$ then $y'' > 0$ similarly



$\begin{matrix} y = 0 \\ y' = 0 \end{matrix} \Rightarrow y \equiv 0$

Theorem

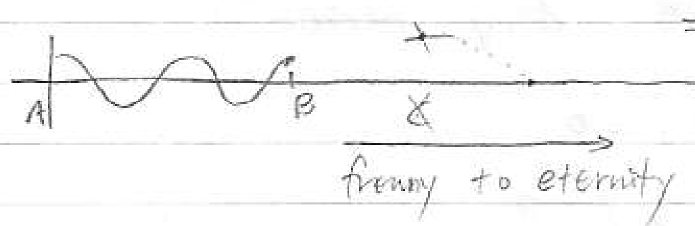
If $q < 0$ on some interval or ray or on $\mathbb{R}$, then on that interval any solution will have at most one zero

Theorem

if $q > 0$ on $[A, \infty)$ & is cont and if $\int_A^\infty q(t)\,dt = \infty$ then any solution y of $y'' + qy = 0$ has infinitely many 0's on $[A, \infty)$

proof

Suppose not



$\Rightarrow \exists B$ s.t on $[B, \infty)$ $y > 0$, $y' > 0$

$$v(x) = -\frac{y'(x)}{y(x)} \quad \text{on } [B, \infty)$$

1. $v < 0$

2. $$v' = -\frac{y''y - y'y'}{y^2}$$
$$= -\frac{-qy\cdot y - y'\cdot y'}{y^2}$$
$$= \frac{qy^2 + (y')^2}{y^2}$$
$$= q + v^2$$

$$\Rightarrow v(x) - v(B) = \int_B^x v'(t)\,dt$$
$$= \int_B^x q(t)\,dt + \int_B^x v^2(t)\,dt$$
$$\geq \int_B^x q\,dt$$

$$v(x) \geq v(B) + \int_B^x q(t)\,dt \longrightarrow +\infty$$

So for large enough x, $v(x) > 0$

contradiction!

$\square$

<u>example</u>

$y'' + xy = 0$. $\quad q = x$

$\int_0^\infty x\,dx = \infty$ Airy oscillates.

So do $y'' + y = 0$

$y'' + \frac{1}{x}y = 0$

$y'' + (\sin x)^2 y = 0$

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What about Bessel's equation (of order α)

$$x^2y'' + \underset{?}{xy'} + (x^2-\alpha^2)y = 0 \quad ?$$

Consider in general $y'' + py' + qy = 0$

Change the "dependent" variable $y = \mu \cdot v$. $\mu(x)$. $v(x)$

$$y'' = \mu''v + 2\mu'v' + \mu v''$$

$$y' = \mu'v + \mu v'$$

$$\mu''v + 2\mu'v' + \mu v'' + p\mu'v + p\mu v' + q\mu v = 0$$

$$\mu v'' + (2\mu' + p\mu)v' + (\mu'' + p\mu' + q\mu)v = 0$$

Find μ s.t. $2\mu' + p\mu = 0$ and then $\mu' = -\frac{p}{2}\mu$

$$\mu v'' + (\mu'' + p\mu' + q\mu)v = 0$$

$$\mu'' = \left(-\frac{p}{2}\mu\right)' = -\frac{p'}{2}\mu - \frac{p}{2}\mu' = -\frac{p'}{2}\mu + \frac{p^2}{4}\mu$$

$$p\mu' = -\frac{p^2}{2}\mu. \qquad q\mu = q\mu$$

$$\mu v'' + \left(q - \frac{p'}{2} - \frac{p^2}{4}\right)\mu v = 0$$

$$\rightsquigarrow v'' + \underbrace{\left(q - \frac{p'}{2} - \frac{p^2}{4}\right)}_{Q}v = 0$$

For Bessel, $p = \frac{1}{x}$, $q = 1 - \frac{\alpha^2}{x^2}$

$$Q = \Big(\underbrace{1 - \frac{\alpha^2}{x^2}}_{q} + \underbrace{\frac{1}{2x^2}}_{-\frac{p'}{2}} - \underbrace{\frac{1}{4x^2}}_{-\frac{p^2}{4}}\Big) = \left(1 + \frac{-\alpha^2 + \frac{1}{4}}{x^2}\right)$$

$$\int_7^\infty Q(t)\,dt = \infty \Rightarrow \text{Bessel oscillates}$$