Reminder: if $f: \mathbb{R} \longrightarrow \mathbb{R}$ $\quad a \in \mathbb{R}$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (\text{if the limit exists})$$

(really meant $f: A \longrightarrow \mathbb{R}$. $a \in A \subset \mathbb{R}$ .. A contains nbhd of a)

Q: How should we define $f(a)$ for $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

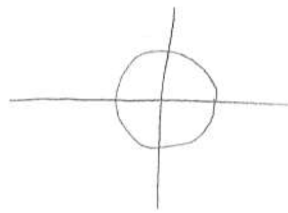
e.g: $f_0 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$f_0(x\ y) = f_0\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} e^x \cos y \\ e^x \sin y \\ (1+x^2+y^2)^{-1} \end{pmatrix}$$

if $f: \mathbb{R} \longrightarrow \mathbb{R}^n$

e.g: $\gamma : \mathbb{R}_t \longrightarrow \mathbb{R}^2_{x,y}$

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$



$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \begin{matrix} \} \mathbb{R}^2 \\ \} \mathbb{R} \end{matrix}$$

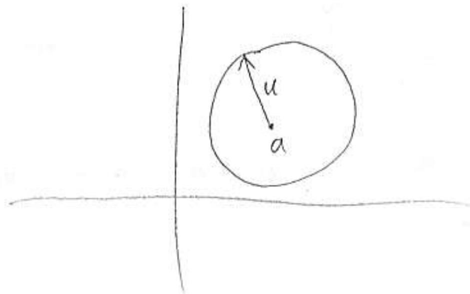
All is well

Check: $\gamma'(t) = \begin{pmatrix} (\cos t)' \\ (\sin t)' \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ naive definition is a total failure

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \begin{matrix} \} \mathbb{R}^m \\ \} \mathbb{R}^n \end{matrix}$$

Attempt : "Directional Derivatives"



$$f'(a;u) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$$

" directional derivative of f at a in the direction of u"

0. $f'(a;0) = 0$

1. $$\underbrace{f'\left(0, \begin{pmatrix}1\\1\end{pmatrix}\right)}_{} ^{\in \mathbb{R}^3} = \lim_{h \to 0} \frac{f_0\left(\begin{pmatrix}0\\0\end{pmatrix} + h\begin{pmatrix}1\\1\end{pmatrix}\right) - f_0\begin{pmatrix}0\\0\end{pmatrix}}{h}$$

$$= \lim_{h \to 0} h^{-1}\left(f_0(h,h) - f_0(0,0) \right)$$

$$= \lim_{h \to 0} h^{-1}\left[\begin{pmatrix} e^h \cos h \\ e^h \sin h \\ (1+2h^2)^{-1} \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} \lim_{h \to 0} \frac{e^h \cos h - e^0 \cos 0}{h} \\ \lim_{h \to 0} \frac{e^h \sin h - e^0 \sin 0}{h} \\ \sim \end{pmatrix}$$

$$= \begin{pmatrix} (e^h \cos h)'_{h=0} \\ (e^h \sin h)'_{h=0} \\ \sim \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ \sim \end{pmatrix}$$

2. $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $a \in \mathbb{R}^n$, $u = e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th row}$

$$f'(a; e_i) = \lim_{h \to 0} h^{-1} \left(f(a + h e_i) - f(a) \right)$$

$$= \lim_{h \to 0} h^{-1} \left(f \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_i + h e_i \\ \vdots \\ a_n \end{pmatrix} - f \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} \right)$$

$$= \frac{d}{dx_i} \begin{pmatrix} \text{the fct you get} \\ \text{from } f \text{ by fixing } x_1, \cdots, x_n \\ \text{to be } a_1, \cdots, a_n \text{ except } x_i \end{pmatrix} (a)$$

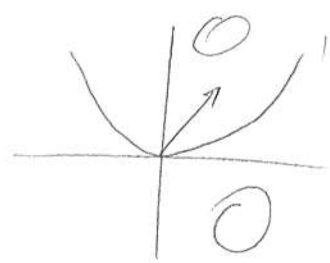
$$= \frac{\partial f}{\partial x_i}(a)$$

e.g: 1. $\dfrac{\partial x^y}{\partial x} = y x^{y-1}$

2. $\dfrac{\partial x^y}{\partial y} = x^y \log x$

e.g: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x, y) = \begin{cases} 1 & y = x^2 \ \& \ x \neq 0 \\ 0 & \text{o.w.} \end{cases}$$



$f'(0; u) = 0$

No chain in dir. der.

$$\frac{f(a+h)-f(a)}{h} \underset{h \to 0}{\approx} f'(a)$$

$$f(a+h)-f(a) \underset{h \to 0}{\approx} f'(a)\cdot h$$

$$\underbrace{f(\overset{\mathbb{R}^n}{a}+\overset{\mathbb{R}^n}{h})}_{\mathbb{R}^m} \underset{\underbrace{h}_{\mathbb{R}^n} \to 0}{\approx} \underbrace{f(a)}_{\mathbb{R}^m} + f'(a)\cdot \underbrace{h}_{\mathbb{R}^n}$$

make sense $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$f'(a)$ should be a lin. trans. $\mathbb{R}^n$ to $\mathbb{R}^m$

($M_{m \times n}$)