Generating Functions

Seq a_k, $k \in \mathbb{Z}_{\geq 0}$ $\Rightarrow$ polynomial / power sums / function

$$F_a(x) = \sum_{k=0}^{\infty/N} a_k x^k$$

$$a_k = \frac{g^{(k)}(0)}{k!} \Longleftarrow g(x)$$

Example

0. $a_k = \frac{1}{k!}$. $\longrightarrow \sum a_k x^k = \sum \frac{x^k}{k!} = e^x$

$$a_k = \begin{cases} \frac{(-1)^{(k-1)/2}}{k!} & k \text{ odd} \\ 0 & k \in 2\mathbb{Z} \end{cases} \longrightarrow \sum a_k x^k = \sin x$$

1. Fix n. $a_k = \binom{n}{k}$

$$F_a(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

2. $a_k = 1$ for all k

$$F_a(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for } |x|<1$$

3.

$$a_k = \begin{cases} 1 & 0 \leq k \leq n \\ 0 & k > n \end{cases}$$

$$F_a(x) = \sum a_k x^k = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

pf:

$$(1-x)\sum_{k=0}^{n} x^k = (1+x+x^2+\cdots+x^n)(1-x)$$
$$= 1-x^{n+1}$$
$$\therefore \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

3.5 $a_k = \alpha^k$

$$F_a(x) = \sum_{k=0}^{\infty} \alpha^k x^k = \sum_{k=0}^{\infty} (\alpha x)^k = \frac{1}{1-\alpha x} \quad \text{when } |\alpha x| < 1 \Leftrightarrow |x| < \frac{1}{|\alpha|}$$

3.6 $1, 2, 4, 8, \ldots \longrightarrow \dfrac{1}{1-2x}$

4 $F_a = (1+x+x^2+x^3)^8 = \cdots$

$$a_k = \frac{1}{k!} F_a^{(k)}(0)$$

$$F_a = \underbrace{(1+x+x^2+x^3)(1+x+x^2+x^3)\cdots(1+x+x^2+x^3)}_{8 \text{ times}}$$

$$\underset{0 \le c_1 \le 3}{x^{c_1}} \qquad \underset{0 \le c_2 \le 3}{x^{c_2}} \qquad \cdots \qquad \underset{0 \le c_8 \le 3}{x^{c_8}}$$

$$= \sum_{0 \le c_i \le 3} x^{c_1 + \cdots + c_8}$$

$$= \sum a_k x^k$$

$$a_k = \#\{(c_1, \cdots c_8) \mid 0 \le c_i \le 3, \ \sum c_i = k\}$$

5. b_k = number of ways of writing k as a sum of 5 non-negative intergers

$$F_b = \underbrace{(1+x+x^2+x^3+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots)}_{5 \text{ times}}$$

$$= \left(\frac{1}{1-x}\right)^5$$

$$= \frac{1}{(1-x)^5}$$

$$= \sum_{k=0}^{\infty} \binom{k+4}{4} x^k$$

In general,

$$\frac{1}{(1-x)^r} = \sum_{k=0}^{\infty} \binom{x+r-1}{r-1} x^k$$

6. given a_k & b_k form $F_a = \sum a_k x^k$, $F_b = \sum b_k x^k$

$g_1 = F_a + F_b$

$g_2 = F_a \cdot F_b$

$g_3 = F_a'$

$g_4 = x F_a'$

which seq does each F_i represents?

$$g_1 = F_a + F_b$$

$$= \sum a_k x^k + \sum b_k x^k$$

$$= \sum_{k=0}^{\infty} (a_k + b_k) x^k \qquad {}_2H_k$$

$$\binom{2+k-1}{k}$$

$$= \sum_{k=0}^{\infty} c_k x^k, \quad c_k = a_k + b_k$$

$$g_2 = \left(\sum a_k x^k\right)\left(\sum b_k x^k\right)$$

$$= (a_0 x^0 + a_1 x^1 + \cdots)(b_0 x^0 + b_1 x^1 + \cdots)$$

$$= \sum_{i,j} a_i x^i \cdot b_j x^j$$

$$= \sum_{i,j} a_i b_j x^{i+j}$$

$$= \sum c_k x^k, \quad c_k = \sum_{i+j=k} a_i b_j = (a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0) = \sum_{i=0}^{k} a_i b_{k-i}$$

$$g_3 = (F_a)'$$

$$= \left(\sum_{k=0}^{\infty} a_k x^k\right)'$$

$$= \sum_{k=0}^{\infty} a_k (x^k)'$$

$$= \sum_{k=1}^{\infty} k a_k x^{k-1} \quad m$$

$$= \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$$

$$= \sum_{m=0}^{\infty} C_m x^m \qquad C_m = (m+1) a_{m+1}$$

$$= F_c$$

$$g_4 = x F_a'$$

$$= \sum_{k=1}^{\infty} k a_k x^k$$

$$= \sum_{k=0}^{\infty} C_k x^k \qquad C_k = k \cdot a_k$$

exercise: ① Compute $\sum_{k=0}^{n} k\binom{n}{k} \rightsquigarrow \sum k\binom{n}{k} x^k$

before $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. $\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n \xrightarrow[x=1]{} 2^n$

If $F = \sum \binom{n}{k} x^k$ then $x \cdot F' = \sum k \binom{n}{k} x^k$

$$\begin{matrix} F = \sum \binom{n}{k} x^k \\ \parallel \\ (1+x)^n \end{matrix}$$

so $\sum k\binom{n}{k} x^k = x \cdot n \cdot (1+x)^{n-1}$

$\Rightarrow$ by putting $x=1$, $\sum k\binom{n}{k} = n \cdot 2^{n-1}$

② $\sum k^2 \binom{n}{k} \longrightarrow \sum k^2 \binom{n}{k} x^k = g$

$$F = \sum \binom{n}{k} x^k = (1+x)^n$$

$$\begin{aligned} g &= x(xF')' \\ &= x\left(x\left((1+x)^n\right)'\right)' \\ &= x\cdot\left(x\cdot n(1+x)^{n-1}\right)' \\ &= n\cdot x\cdot\left(x(1+x)^{n-1}\right)' \\ &= n\cdot x\cdot\left((1+x)^{n-1} + (n-1)x(1+x)^{n-2}\right) \end{aligned}$$

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 2 && 1 && \\ & 1 && 3 && 3 && 1 & \\ n=4 \quad 1 && 4 && 6 && 4 && 1 \end{array}$$

$\rightarrow 0+4+24+36+16=80$

$\xrightarrow{x=1}$ $n\cdot\left(2^{n-1} + (n-1)\cdot 2^{n-2}\right)$

$\xrightarrow{n=4}$ $4(8+3\cdot 4) = 80$

③ How many ways to select "k" toys from 7 types of toys with 2~6 chosen of each type?

$$F_a = \sum a_k x^k = \underbrace{(x^2+x^3+\cdots+x^6)(x^2+\cdots+x^6)\cdots(x^2+\cdots+x^6)}_{7 \text{ times.}}$$

C_1 : # rubber duckies

C_2 : # balls

C_3 : # barbies

$\Bigg\} \rightarrow \sum_{i=1}^{7} C_i = k$

$$\begin{aligned} F_a &= (x^2+\cdots+x^6)^7 \\ &= \left(x^2(1+x+\cdots+x^4)\right)^7 \\ &= x^{14}\left(\frac{1-x^5}{1-x}\right)^7 \\ &= x^{14}\cdot(1-x^5)^7\cdot\frac{1}{(1-x)^7} \end{aligned}$$

$\frac{1}{(1-x)^r} = \sum \binom{x+r-1}{k} x^r \longrightarrow = x^{14}\cdot(1-x^5)^7\left(\sum \binom{x+6}{6} x^k\right)$

What's the coeff. of x^{25} in the above?

a_{25} = the coeff. of x^{11} in $(1-x^5)^7 \sum_{k=0}^{\infty} \binom{k+6}{6} x^k$

= " $\left(1-7x^5+\binom{7}{2}x^{10}-\cdots x^{15}\right)\left(\sum \binom{x+6}{6} x^k\right)$

$= \binom{11+6}{6} - 7\binom{6+6}{6} + \binom{7}{2}\binom{1+6}{6}$