Pensieve header: Examples for the Da-Nang talk: Double Integration and the trefoil.

Startup

```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\DaNang-1905"];
<< "Engine-Speedy.m";
<< "Objects.m";
```

cm

```
In[•]:= Λ0 = HoldForm[(η_i + e^(-α_i-ϵ β_i) η_j/(1+ϵ η_j ξ_i)) y_k + (β_i + β_j + Log[1+ϵ η_j ξ_i]/ϵ) b_k +
          (α_i + α_j + Log[1+ϵ η_j ξ_i]) a_k + (e^(-α_j-ϵ β_j) ξ_i/(1+ϵ η_j ξ_i) + ξ_j) x_k];
TeXForm[Λ0]
Λ = ReleaseHold[Λ0]
```

Out[•]=
$$a_k \left(\text{Log}[1+\epsilon \eta_j \xi_i]+\alpha_i+\alpha_j\right)+b_k \left(\frac{\text{Log}[1+\epsilon \eta_j \xi_i]}{\epsilon}+\beta_i+\beta_j\right)+y_k \left(\eta_i+\frac{e^{-\alpha_i-\epsilon \beta_i} \eta_j}{1+\epsilon \eta_j \xi_i}\right)+x_k \left(\frac{e^{-\alpha_j-\epsilon \beta_j} \xi_i}{1+\epsilon \eta_j \xi_i}+\xi_j\right)$$

```
\left(\eta _i+\frac{e^{-\alpha _i-\epsilon  \beta _i} \eta _j}{1+\epsilon  \eta _j \xi _i}
   y_k+\left(\beta _i+\beta _j+\frac{\log \left(1+\epsilon  \eta _j \xi _i\right)}{\epsilo
   }\right) b_k+\left(\alpha _i+\alpha _j+\log \left(1+\epsilon  \eta _j \xi _i\right)\rig
   a_k+\left(\frac{e^{-\alpha _j-\epsilon  \beta _j} \xi _i}{1+\epsilon  \eta _j \xi _i}+\
   _j\right) x_k
```

rho

```
In[•]:= HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, , ]];
{ŷ = ((0 0),(ϵ 0)), b̂ = ((0 0),(0 -ϵ)), â = ((1 0),(0 0)), x̂ = ((0 1),(0 0))};
HL /@ {â.x̂ - x̂.â == x̂, â.ŷ - ŷ.â == -ŷ, b̂.ŷ - ŷ.b̂ == -ϵ ŷ, b̂.x̂ - x̂.b̂ == ϵ x̂, x̂.ŷ - ŷ.x̂ == b̂ + ϵ â}
```

rho

Out[•]= {True, True, True, True, True}

rho

```
In[•]:= HL@Simplify@With[{𝔼 = MatrixExp},
  𝔼[η_i ŷ].𝔼[β_i b̂].𝔼[α_i â].𝔼[ξ_i x̂].𝔼[η_j ŷ].𝔼[β_j b̂].𝔼[α_j â].𝔼[ξ_j x̂] ==
   𝔼[ŷ ∂_(y_k) Λ].𝔼[b̂ ∂_(b_k) Λ].𝔼[â ∂_(a_k) Λ].𝔼[x̂ ∂_(x_k) Λ]]
```

rho

Out[•]= True

rho

```
In[•]:= Series[Λ, {ϵ, 0, 1}]
```

rho

Out[•]=
$$\left(a_k \left(\alpha_i+\alpha_j\right)+y_k \left(\eta_i+e^{-\alpha_i} \eta_j\right)+b_k \left(\beta_i+\beta_j+\eta_j \xi_i\right)+x_k \left(e^{-\alpha_j} \xi_i+\xi_j\right)\right)+\left(a_k \eta_j \xi_i-\frac{1}{2} b_k \eta_j^2 \xi_i^2-e^{-\alpha_i} y_k \eta_j \left(\beta_i+\eta_j \xi_i\right)-e^{-\alpha_j} x_k \xi_i \left(\beta_j+\eta_j \xi_i\right)\right) \epsilon+O[\epsilon]^2$$

Some Atoms

Atoms

In[•]:=
```
PP_ := Identity; $k = 1; ħ = γ = 1;
Column[{(# → (ε = ToExpression[#]; Normal@Simplify[ε[[1]] + ε[[2]] + Log@ε[[3]]])) & /@
  {"dm_{i,j→k}", "dΔ_{i→j,k}", "dS_i", "R_{i,j}", "P_{i,j}"}}]
```

Atoms

$$dm_{i,j\to k} \to a_k\ (\alpha_i+\alpha_j) + b_k\ (\beta_i+\beta_j) + y_k\ \eta_i + \frac{y_k\ \eta_j}{\mathcal{A}_i} + \frac{x_k\ \xi_i}{\mathcal{A}_j} + \eta_j\ \xi_i - B_k\ \eta_j\ \xi_i + \frac{\in\left(2\,y_k\,\eta_j\,(2\,x_k\,\xi_i+\mathcal{A}_j\,(-2\,\beta_i+(1-3\,B_k)\,\eta_j\,\xi_i))+\mathcal{A}_i\,\xi_i\,\left(x_k\,(-4\,\beta_j+2\,(1-3\,B_k)\,\eta_j\,\xi_i)+\mathcal{A}_j\,\eta_j\,\left(4\,a_k\,B_k+\left(1-4\,B_k+3\,B_k^2\right)\,\eta_j\,\xi_i\right)\right)\right)}{4\,\mathcal{A}_i\,\mathcal{A}_j} + x_k\ \xi_j$$

$$d\Delta_{i\to j,k} \to a_j\ \alpha_i + a_k\ \alpha_i + b_j\ \beta_i + b_k\ \beta_i + y_j\ \eta_i + B_j\ y_k\ \eta_i + x_j\ \xi_i + x_k\ \xi_i + \frac{1}{2}\in\left(B_j\ y_j\ y_k\ \eta_i^2 + x_k\ \xi_i\ \left(-2\,a_j + x_j\ \xi_i\right)\right)$$

Out[•]=
$$dS_i \to -a_i\ \alpha_i - b_i\ \beta_i - \frac{\mathcal{A}_i\ (y_i\ \eta_i+(-\eta_i+B_i\ (x_i+\eta_i))\ \xi_i)}{B_i} - \frac{1}{4\,B_i^2}\in\mathcal{A}_i\ \left(\mathcal{A}_i\ \eta_i^2\ \left(2\,y_i^2 - 6\,y_i\ \xi_i + 3\,\xi_i^2\right) + B_i^2\ \xi_i\ \left(4\,a_i\,x_i + 2\,x_i^2\,\mathcal{A}_i\ \xi_i + 2\,x_i\ \left(2\,\beta_i + \mathcal{A}_i\ \eta_i\ \xi_i\right) + \eta_i\ (-4 + 4\,\beta_i + \mathcal{A}_i\ \eta_i\ \xi_i)\right) + 2\,B_i\ \eta_i\ \left(y_i\ \left(-2 + 2\,\beta_i + 2\,x_i\,\mathcal{A}_i\ \xi_i + \mathcal{A}_i\ \eta_i\ \xi_i\right) - \xi_i\ \left(-2 + 2\,a_i + 2\,\beta_i + 3\,x_i\,\mathcal{A}_i\ \xi_i + 2\,\mathcal{A}_i\ \eta_i\ \xi_i\right)\right)\right)$$

$$R_{i,j} \to a_j\ b_i + x_j\ y_i - \frac{1}{4}\in x_j^2\ y_i^2$$

$$P_{i,j} \to \alpha_j\ \beta_i + \eta_i\ \xi_j + \frac{1}{4}\in\eta_i^2\ \xi_j^2$$

Double Integration

Integrals

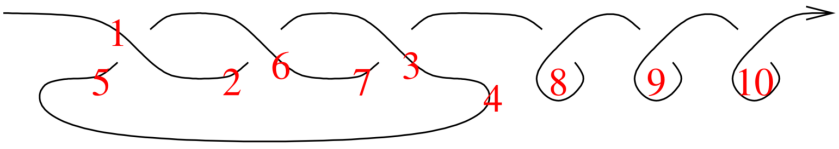
In[•]:=
```
inp = 𝔼_{}→{1}[3 a_1 b_1, 5 x_1 y_1, 1] // dm_{i,1→i};
Table[
  HL@TrueQ[
     (inp // (sϒ_{i→1,1,2,2} RR) // BM // AM // P_{1,2}) dϵ_j ≡
      (inp // ΔΔ // (sϒ_{i→1,1,2,2} RR) // BM // AM // P_{1,2})],
  {ΔΔ, {dΔ_{i→i,j}, dΔ_{i→j,i}}}, {AM, {dm_{2,4→2}, dm_{4,2→2}}}, {BM, {dm_{1,3→1}, dm_{3,1→1}}},
  {RR, {R_{3,4}, R_{3,4} // dS_3 // dS_3, R_{3,4} // dS_4 // dS_4}}
 ] // MatrixForm
```

Out[•]//MatrixForm=
Integrals

False False False False False False	False False True False False False
False False False False False True	False False False False False False

The Trefoil

Trefoil



Trefoil

In[•]:=
```
$k = 2;
Simplify[R_{1,5} R_{6,2} R_{3,7} C̄_4 K̄ink_8 K̄ink_9 K̄ink_10 // dm_{1,2→1} // dm_{1,3→1} // dm_{1,4→1} // dm_{1,5→1} // dm_{1,6→1} //
    dm_{1,7→1} // dm_{1,8→1} // dm_{1,9→1} // dm_{1,10→1}] /. v_1 :→ v
```

Trefoil

Out[•]=
$$\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{B}{1-B+B^2}+\frac{B\left(-B+2B^2+2B^4+a\left(-1+B-B^3+B^4\right)-2xy-B^3\left(3+2xy\right)\right)\in}{\left(1-B+B^2\right)^3}+\frac{1}{2\left(1-B+B^2\right)^5}$$
$$B\Big(4B^8+a^2\left(1-B+B^2\right)^2\left(1+B-6B^2+B^3+B^4\right)+6B^5x^2y^2+2xy\left(-2+3xy\right)-B^7\left(11+4xy\right)-$$
$$2B^2\left(1+6x^2y^2\right)-2B^4\left(1-2xy+6x^2y^2\right)+B\left(1+8xy+6x^2y^2\right)+B^6\left(6+8xy+6x^2y^2\right)+$$
$$B^3\left(4+4xy+30x^2y^2\right)+2a\left(1-B+B^2\right)\left(2B^6+2xy+8B^3\left(1+xy\right)-$$
$$5B^2\left(1+2xy\right)-2B^5\left(1+2xy\right)-B^4\left(7+2xy\right)+B\left(2+4xy\right)\right)\Big)\in^2+O[\in]^3\Big]$$