The Strongest Genuinely Computable Knot Invariant in 2024

Abstract. "Gennuinely computable" means we have computed it for random knots with over 300 crossings."Strongest" means it separates prime knots with up to 15 crossings better than the less-computable HOMFLY-PT and Khovanov homology taken together. And hey, it's also meaningful and fun.

Continues Rozansky, Garoufalidis, Kricker, and Ohtsuki, joint with van der Veen.

van der Veen

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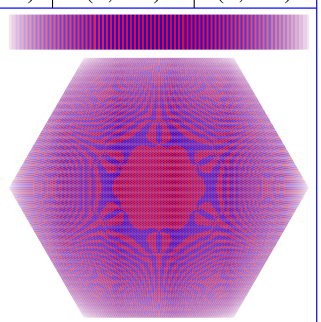
Strongest. Testing $\Theta = (\Delta, \theta)$ on prime knots up to mirrors and reversals, counting the number of distinct values (with deficits in parenthesis): (ρ_1: [Ro1, Ro2, Ro3, Ov, BV1])

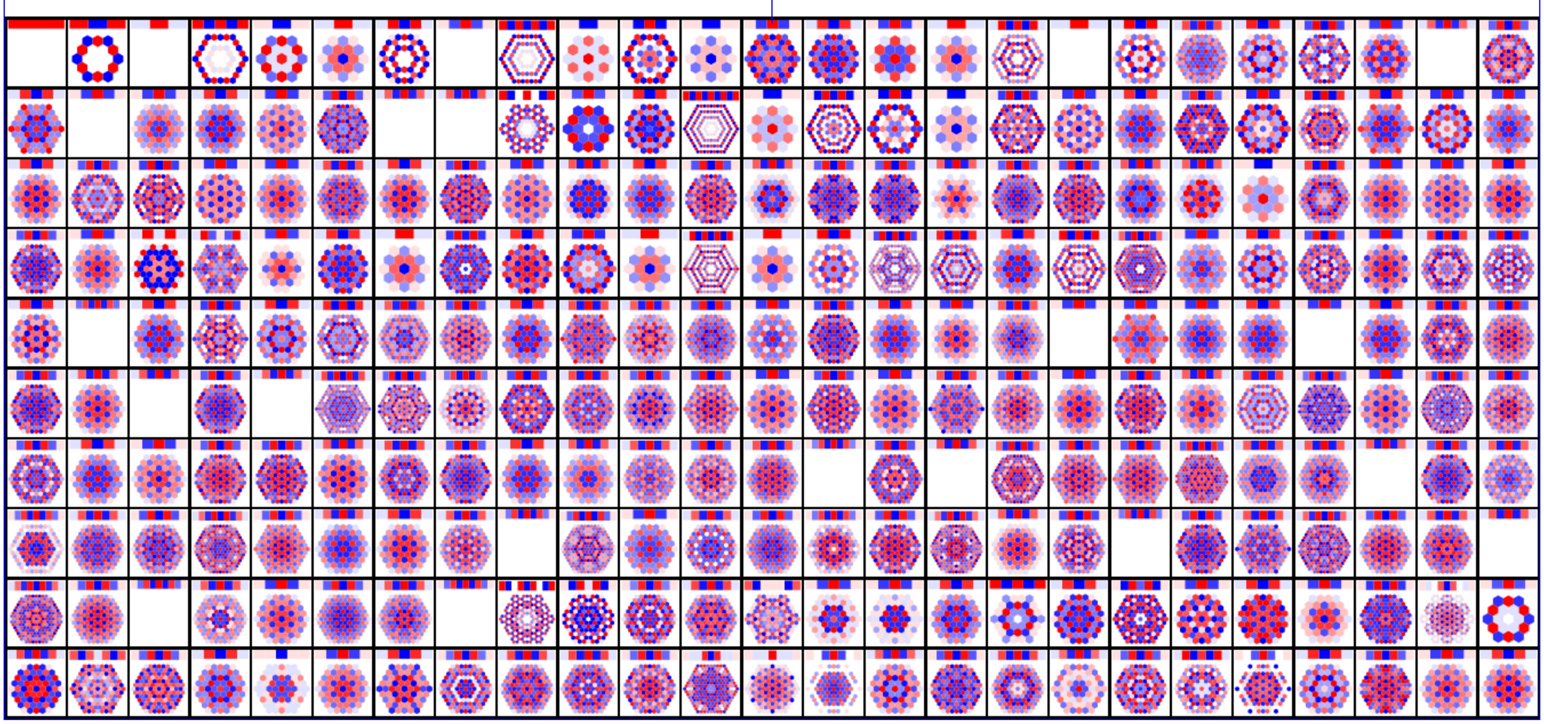
	knots	(H, Kh)	(Δ, ρ_1)	$\Theta = (\Delta, \theta)$	together
reign		2005-22	2022-24	2024-	
xing ≤ 10	249	248 (1)	249 (0)	249 (0)	249 (0)
xing ≤ 11	801	771 (30)	787 (14)	798 (3)	798 (3)
xing ≤ 12	2,977	(214)	(95)	(19)	(18)
xing ≤ 13	12,965	(1,771)	(959)	(194)	(185)
xing ≤ 14	59,937	(10,788)	(6,253)	(1,118)	(1,062)
xing ≤ 15	313,230	(70,245)	(42,914)	(6,758)	(6,555)

Genuinely Computable. Here's Θ on a random 300 crossing knot (from [DHOEBL]). For almost every other invariant, that's science fiction.



Fun. There's so much more to see in 2D pictutres than in 1D ones! Yet almost nothing of the patterns you see we know how to prove. We'll have fun with that over the next few years. Would you join?

Meaningful. Θ gives a genus bound (unproven yet with confidence). We hope (with reason) it says something about ribbon knots.



Questions, Conjectures, Expectations, Dreams.

Question 1. What's the relationship between Θ and the Garoufalidis-Kashaev invariants [GK, GL]?

Conjecture 2. On classical (non-virtual) knots, θ always has hexagonal (D_6) symmetry.

Conjecture 3. θ is the ϵ^1 contribution to the "solvable approximation" of the sl_3 universal invariant, obtained by running the quantization machinary on the double $\mathcal{D}(\mathfrak{b}, b, \epsilon\delta)$, where $\mathfrak{b}$ is the Borel subalgebra of sl_3, b is the bracket of $\mathfrak{b}$, and δ the cobracket. See [BV2, BN1, Sch]

Conjecture 4. θ is equal to the "two-loop contribution to the Kontsevich Integral", as studied by Garoufalidis, Rozansky, Kricker, and in great detail by Ohtsuki [GR, Ro1, Ro2, Ro3, Kr, Oh].

Fact 5. θ has a perturbed Gaussian integral formula, with integration carried out over over a space $6E$, consisting of 6 copies of the space of edges of a knot diagram D. See [BN2].

Conjecture 6. For any knot K, its genus $g(K)$ is bounded by the T_1-degree of θ: $g(K) < \lceil \deg_{T_1} \theta(K) \rceil$.

Conjecture 7. $\theta(K)$ has another perturbed Gaussian integral formula, with integration carried out over over the space $6H_1$, consisting of 6 copies of $H_1(\Sigma)$, where Σ is a Seifert surface for K.

Expectation 8. There are many further invariants like θ, given by Green function formulas and/or Gaussian integration formulas. One or two of them may be stronger than θ and as computable.

Dream 9. These invariants can be explained by something less foreign than semisimple Lie algebras.

Dream 10. θ will have something to say about ribbon knots.

References.

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Preliminaries

This is Theta.nb of http://drorbn.net/ubc24/ap.

```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

```
C:\drorbn\AcademicPensieve\Projects\KnotTheory\KnotTheory
```

```
Loading KnotTheory` version
   of September 27, 2024, 13:23:33.5336.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from http://drorbn.net/ubc24/ap
   to compute rotation numbers.
```

```
Loading PolyPlot.m from http://drorbn.net/ubc24/ap
   to plot 2-variable polynomials.
```

The Program

```
CF[ℰ_] :=
  Module[{vs = Union@Cases[ℰ, g__, ∞], ps, c},
   Total[CoefficientRules[Expand[ℰ], vs] /.
     (ps_ → c_) ⧴ Factor[c] (Times @@ vs^ps)]];
```

```
T3 = T1 T2;
```

```
R1[s_, i_, j_] =
  CF[
   s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj -
       (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
      g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
       ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
         (T3^s - 1) g3ji
          (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji +
            (T2^s - 2) g2jj + g2ij)) / (T2^s - 1))];
```

```
θ[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
 CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1
   ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
```

```
Γ1[φ_, k_] = -φ/2 + φ g3kk;
```

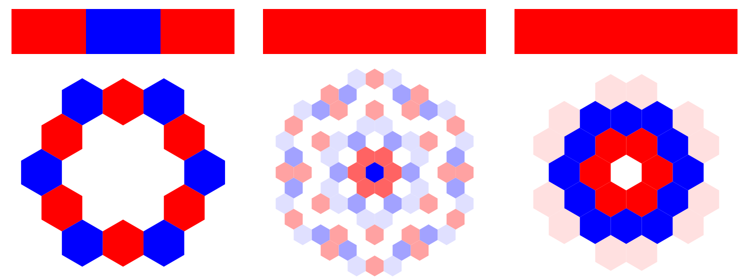
```
Θ[K_] :=
  Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ν, α,
    β, gEval, c, z},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} ⧴
     (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
   Δ = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
   G = Inverse[A];
   gEval[ℰ_] :=
    Factor[ℰ /. g_ν_,α_,β_ ⧴ (G[[α, β]] /. T → T_ν)];
   z = gEval[Sum_{k1=1}^n Sum_{k2=1}^n θ[Cs[[k1]], Cs[[k2]]]];
   z += gEval[Sum_{k=1}^n R1 @@ Cs[[k]]];
   z += gEval[Sum_{k=1}^{2 n} Γ1[φ[[k]], k]];
   {Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) z} //
    Factor];
```

The Trefoil, Conway, and Kinoshita-Terasaka

```
Θ[Knot[3, 1]] // Expand
```

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2^2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}$$

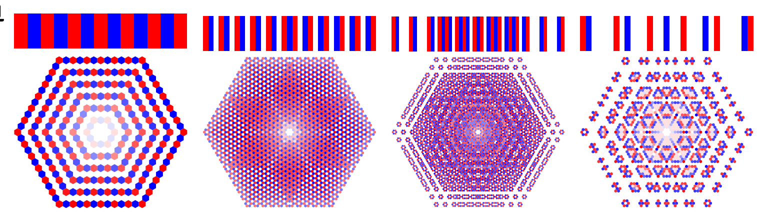
```
GraphicsRow[PolyPlot[Θ[Knot[#]]] & /@
  {"3_1", "K11n34", "K11n42"}]
```



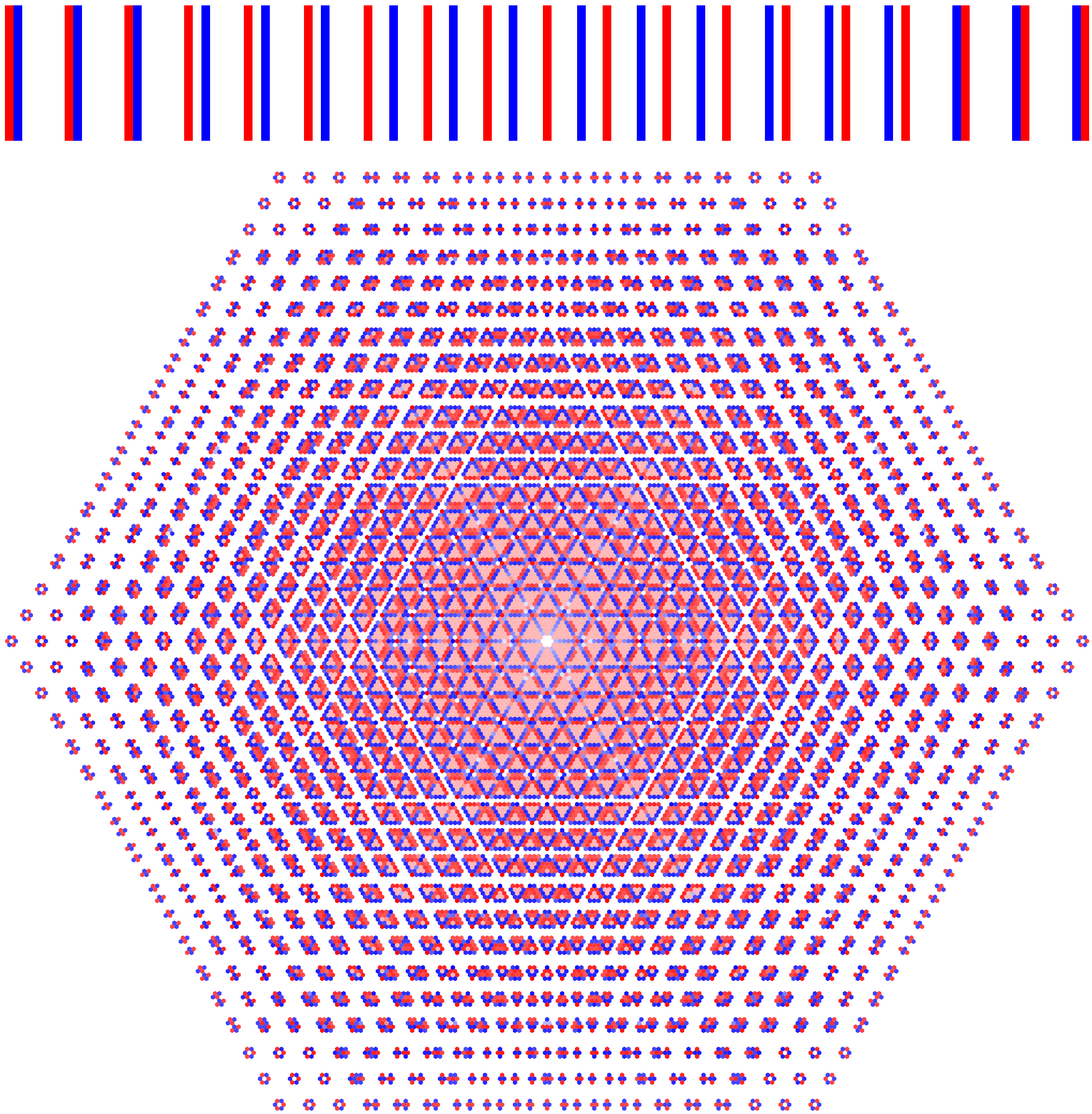
(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

Some Torus Knots

```
GraphicsRow[PolyPlot[Θ[TorusKnot @@ #]] &
  /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}},
 Spacings → Scaled@0.05]
```



The torus knot $T_{22/7}$: (many more at ωεβ/TK)



Random knots from [DHOEBL], with 50-73 crossings: (many more at ωεβ/DK)