Pensieve header: The full sl_2 invariant using the Drinfel'd double. Based on Projects/SL2Invariant/SL2Invariant.nb.

Program

Program

Utilities

In[∘]:=
```
$k = 2; (*ℏ=γ=1;*)
```

Canonical Form:

Program

In[∘]:=
```
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
    Expand[ℰ] //. e^x_ e^y_ :> e^(x+y) /. e^x_ :> e^CF[x]];
```

The Kronecker δ:

Program

In[∘]:=
```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

Program

In[∘]:=
```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Program

Zip and Bind

Variables and their duals:

Program

In[∘]:=
```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)_i;
```

Finite Zips:

Program

In[∘]:=
```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip_{}[P_] := P; Zip_{ζ_,ζs___}[P_] :=
 (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :> ∂_{ζ*,d} f) /. ζ* → 0
```

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

Program

In[∘]:=
```
QZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
  zs = Table[ζ*, {ζ, ζs}];
  c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
  ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
  ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
  qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
  zrule = Thread[zs → qt.(zs + ys)];
  Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
  CF /@ 𝔼[L, Q2, Det[qt] 𝕖^(-Q2) Zip_{ζs}[𝕖^Q1 (P /. zrule)]] ];
```

Upper to lower and lower to Upper:

Program

In[∘]:=
```
U2l = {B_{i_}^{p_.} → 𝕖^(-p ℏ γ b_i), B^{p_.} → 𝕖^(-p ℏ γ b), T_{i_}^{p_.} → 𝕖^(p ℏ t_i), T^{p_.} → 𝕖^(p ℏ t), 𝒜_{i_}^{p_.} → 𝕖^(p γ α_i), 𝒜^{p_.} → 𝕖^(p γ α)};
l2U = {𝕖^(c_. b_{i_} + d_.) :→ B_i^(-c/(ℏ γ)) 𝕖^d, 𝕖^(c_. b + d_.) :→ B^(-c/(ℏ γ)) 𝕖^d,
  𝕖^(c_. t_{i_} + d_.) :→ T_i^(c/ℏ) 𝕖^d, 𝕖^(c_. t + d_.) :→ T^(c/ℏ) 𝕖^d,
  𝕖^(c_. α_{i_} + d_.) :→ 𝒜_i^(c/γ) 𝕖^d, 𝕖^(c_. α + d_.) :→ 𝒜^(c/γ) 𝕖^d,
  𝕖^ℰ_ :→ 𝕖^(Expand@ℰ)};
```

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = \mathrm{P}e^{L+Q}$. Such zips regard all of $\mathrm{P}e^{Q}$ as a single”P”. Here the z’s are b and α and the ζ’s are β and a.

Program

In[∘]:=
```
LZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
  zs = Table[ζ*, {ζ, ζs}];
  c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
  ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
  ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
  lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
  zrule = Thread[zs → lt.(zs + ys)];
  L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
  Q2 = (Q1 = Q /. U2l /. zrule) /. Alternatives @@ zs → 0;
  CF /@ 𝔼[L2, Q2, Det[lt] 𝕖^(-L2-Q2) Zip_{ζs}[𝕖^(L1+Q1) (P /. U2l /. zrule)]] //. l2U ];
```

Program

In[∘]:=
```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := Module[{n}, Times[
    L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
    R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
   ] // LZip_{Join@@Table[{β_{n@i}, τ_{n@i}, a_{n@i}}, {i, {is}}]} // QZip_{Join@@Table[{ξ_{n@i}, y_{n@i}}, {i, {is}}]} ];
B_{is___}[L_, R_] := B_{{is}}[L, R];
```

Program

𝔼 morphisms with domain and range.

Program

In[∘]:=
```
B_{is_List}[𝔼_{d1_→r1_}[L1_, Q1_, P1_], 𝔼_{d2_→r2_}[L2_, Q2_, P2_]] :=
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ B_is[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] // 𝔼_{d2_→r2_}[L2_, Q2_, P2_] :=
  B_{r1⋂d2}[𝔼_{d1→r1}[L1, Q1, P1], 𝔼_{d2→r2}[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] ≡ 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_{d1_→r1_}[L1_, Q1_, P1_] 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼_{d_→r_}[L_, Q_, P_]_{$k_} := 𝔼_{d→r} @@ 𝔼[L, Q, P]_{$k};
𝔼_[ℰ___][i_] := {ℰ}〚i〛;
```

Program

“Define” code

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

Program

In[∘]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
```

Program

In[∘]:=
```
Define[op_{is__} = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
     SD[op_{nisp,$k_Integer}, Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
     SD[op_{isp}, op_{{is},$k}]; SD[op_{sis__}, op_{{sis}}];
    ] /. {SD → SetDelayed,
     isp → {is} /. {i → i_, j → j_, k → k_},
     nis → {is} /. {i → ii, j → jj, k → kk},
     nisp → {is} /. {i → ii_, j → jj_, k → kk_}
    }] ]]
```

Program

The Fundamental Tensors

Program

In[∘]:=
```
Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k, (e^{-γ α_j} ξ_i + ξ_j) x_k, 1]_{$k},
 bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k, (η_i + η_j) y_k, e^{(e^{-ϵ β_i}-1) η_j y_k}]_{$k}]
```

Program

In[∘]:=
```
Define[R_{i,j} =
  𝔼_{{}→{i,j}}[ℏ a_j b_i, ℏ x_j y_i, e^(Sum[(1 - e^{γ ϵ ℏ})^k (ℏ y_i x_j)^k / (k (1 - e^{k γ ϵ ℏ})), {k, 2, $k+1}])]_{$k}]
```

Program

In[∘]:=
```
Define[R̄i,j = 𝔼{}→{i,j}[-ℏ aj bi, -ℏ xj yi / Bi,
    1 + If[$k == 0, 0, (R̄{i,j},$k-1)$k[3] -
      (((R̄{i,j},0)$k R1,2 (R̄{3,4},$k-1)$k) // (bmi,1→i amj,2→j) // (bmi,3→i amj,4→j))[3]]],
  Pi,j = 𝔼{i,j}→{}[βi αj / ℏ, ηi ξj / ℏ, 1 + If[$k == 0, 0, (P{i,j},$k-1)$k[3] -
      (R1,2 // ((P{1,j},0)$k (P{i,2},$k-1)$k))[3]]]]
```

Program

In[∘]:=
```
Define[aSj = R̄i,j ~ Bi ~ Pi,j,
  aS̄i = 𝔼{i}→{i}[-ai αi, -xi 𝒜i ξi, 1 + If[$k == 0, 0, (aS̄{i},$k-1)$k[3] -
      ((aS̄{i},0)$k ~ Bi ~ aSi ~ Bi ~ (aS̄{i},$k-1)$k)[3]]]]
```

Program

In[∘]:=
```
Define[bSi = Ri,1 ~ B1 ~ aS1 ~ B1 ~ Pi,1,
  bS̄i = Ri,1 ~ B1 ~ aS̄1 ~ B1 ~ Pi,1,
  aΔi→j,k = (R1,j R2,k) // bm1,2→3 // P3,i,
  bΔi→j,k = (Rj,1 Rk,2) // am1,2→3 // Pi,3]
```

Program

In[∘]:=
```
Define[dmi,j→k = (𝔼{i,j}→{i,j}[βi bi + αj aj, ηi yi + ξj xj, 1]
      (aΔi→1,2 // aΔ2→2,3 // aS̄3) (bΔj→-1,-2 // bΔ-2→-2,-3)) // (P-1,3 P-3,1 am2,j→k bmi,-2→k),
  dSi = 𝔼{i}→{1,2}[βi b1 + αi a2, ηi y1 + ξi x2, 1] // (bS̄1 aS2) // dm2,1→i,
  dΔi→j,k = (bΔi→3,1 aΔi→2,4) // (dm3,4→k dm1,2→j)]
```

Program

In[∘]:=
```
Define[Ci = 𝔼{}→{i}[0, 0, Bi^(1/2) ⅇ^(-ℏ ϵ ai/2)]$k,
  C̄i = 𝔼{}→{i}[0, 0, Bi^(-1/2) ⅇ^(ℏ ϵ ai/2)]$k,
  Kinki = (R1,3 C̄2) // dm1,2→1 // dm1,3→i,
  K̄inki = (R̄1,3 C2) // dm1,2→1 // dm1,3→i]
```

Note. $t == \epsilon \text{a} - \gamma \text{b}$ and $b == -t/\gamma + \epsilon \text{a}/\gamma$.

Program

In[∘]:=
```
Define[b2ti = 𝔼{i}→{i}[αi ai - βi ti / γ, ξi xi + ηi yi, ⅇ^(ϵ βi ai/γ)]$k,
  t2bi = 𝔼{i}→{i}[αi ai - τi γ bi, ξi xi + ηi yi, ⅇ^(ϵ τi ai)]$k]
```

Program

In[∘]:=
```
Define[kRi,j = Ri,j // (b2ti b2tj) /. ti|j → t,
  kR̄i,j = R̄i,j // (b2ti b2tj) /. {ti|j → t, Ti|j → T},
  kmi,j→k = (t2bi t2bj) // dmi,j→k // b2tk /. {tk → t, Tk → T, τi|j → 0},
  kCi = Ci // b2ti /. Ti → T, kC̄i = C̄i // b2ti /. Ti → T,
  kKinki = Kinki // b2ti /. {ti → t, Ti → T},
  kK̄inki = K̄inki // b2ti /. {ti → t, Ti → T}]
```

Testing

In[ ]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

In[∘]:=

$$\mathbf{Block}\Big[\{\$k = 1\},\ \{\mathbf{am} \to \mathbf{am}_{i,j\to k},\ \mathbf{bm} \to \mathbf{bm}_{i,j\to k},\ \mathbf{dm} \to \mathbf{dm}_{i,j\to k},\ \mathbf{R} \to \mathbf{R}_{i,j},\ \overline{\mathbf{R}} \to \overline{\mathbf{R}}_{i,j},\ \mathbf{P} \to \mathbf{P}_{i,j},$$
$$\mathbf{aS} \to \mathbf{aS}_i,\ \overline{\mathbf{aS}} \to \overline{\mathbf{aS}}_i,\ \mathbf{bS} \to \mathbf{bS}_i,\ \overline{\mathbf{bS}} \to \overline{\mathbf{bS}}_i,\ \mathbf{dS} \to \mathbf{dS}_i,\ \mathbf{a\Delta} \to \mathbf{a\Delta}_{i\to j,k},\ \mathbf{b\Delta} \to \mathbf{b\Delta}_{i\to j,k},$$
$$\mathbf{d\Delta} \to \mathbf{d\Delta}_{i\to j,k},\ \mathbf{C} \to \mathbf{C}_i,\ \overline{\mathbf{C}} \to \overline{\mathbf{C}}_i,\ \mathbf{Kink} \to \mathbf{Kink}_i,\ \overline{\mathbf{Kink}} \to \overline{\mathbf{Kink}}_i,\ \mathbf{b2t} \to \mathbf{b2t}_i,\ \mathbf{t2b} \to \mathbf{t2b}_i$$
$$\}\Big]\ //\ \mathbf{Column}$$

Out[∘]=

$$am \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,(\alpha_i + \alpha_j),\ x_k\left(e^{-\gamma\,\alpha_j}\,\xi_i + \xi_j\right),\ 1\right]$$

$$bm \to \mathbb{E}_{\{i,j\}\to\{k\}}\left[b_k\,(\beta_i + \beta_j),\ y_k\,(\eta_i + \eta_j),\ 1 - y_k\,\beta_i\,\eta_j\,\epsilon + O[\epsilon]^2\right]$$

$$dm \to \mathbb{E}_{\{i,j\}\to\{k\}}\Big[a_k\,\alpha_i + a_k\,\alpha_j + b_k\,\beta_i + b_k\,\beta_j,\ \frac{1}{\hbar\,\mathcal{A}_i\,\mathcal{A}_j}$$
$$(\hbar\,y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_i + \hbar\,y_k\,\mathcal{A}_j\,\eta_j + \hbar\,x_k\,\mathcal{A}_i\,\xi_i + \mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i - B_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i + \hbar\,x_k\,\mathcal{A}_i\,\mathcal{A}_j\,\xi_j),$$
$$1 + \frac{1}{4\,\hbar\,\mathcal{A}_i\,\mathcal{A}_j}\Big(-4\,\hbar\,y_k\,\mathcal{A}_j\,\beta_i\,\eta_j - 4\,\hbar\,x_k\,\mathcal{A}_i\,\beta_j\,\xi_i + 4\,\gamma\,\hbar^2\,x_k\,y_k\,\eta_j\,\xi_i +$$
$$4\,\hbar\,a_k\,B_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i + 2\,\gamma\,\hbar\,y_k\,\mathcal{A}_j\,\eta_j^2\,\xi_i - 6\,\gamma\,\hbar\,B_k\,y_k\,\mathcal{A}_j\,\eta_j^2\,\xi_i + 2\,\gamma\,\hbar\,x_k\,\mathcal{A}_i\,\eta_j\,\xi_i^2 -$$
$$6\,\gamma\,\hbar\,B_k\,x_k\,\mathcal{A}_i\,\eta_j\,\xi_i^2 + \gamma\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2 - 4\,\gamma\,B_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2 + 3\,\gamma\,B_k^2\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j^2\,\xi_i^2\Big)\,\epsilon + O[\epsilon]^2\Big]$$

$$R \to \mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,b_i,\ \hbar\,x_j\,y_i,\ 1 - \frac{1}{4}\left(\gamma\,\hbar^3\,x_j^2\,y_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{R} \to \mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,b_i,\ -\frac{\hbar\,x_j\,y_i}{B_i},\ 1 - \frac{\left(4\,\hbar^2\,a_j\,B_i\,x_j\,y_i + 3\,\gamma\,\hbar^3\,x_j^2\,y_i^2\right)\epsilon}{4\,B_i^2} + O[\epsilon]^2\right]$$

$$P \to \mathbb{E}_{\{i,j\}\to\{\}}\left[\frac{\alpha_j\,\beta_i}{\hbar},\ \frac{\eta_i\,\xi_j}{\hbar},\ 1 + \frac{\gamma\,\eta_i^2\,\xi_j^2\,\epsilon}{4\,\hbar} + O[\epsilon]^2\right]$$

$$aS \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1 + \frac{1}{2}\left(-2\,\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i - \gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{aS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1 + \frac{1}{2}\left(2\,\gamma\,\hbar\,x_i\,\mathcal{A}_i\,\xi_i - 2\,\hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i - \gamma\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$bS \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1 + \frac{\left(-2\,B_i\,y_i\,\beta_i\,\eta_i - \gamma\,\hbar\,y_i^2\,\eta_i^2\right)\epsilon}{2\,B_i^2} + O[\epsilon]^2\right]$$

$$\overline{bS} \to \mathbb{E}_{\{i\}\to\{i\}}\left[-b_i\,\beta_i,\ -\frac{y_i\,\eta_i}{B_i},\ 1 + \frac{\left(2\,\gamma\,\hbar\,B_i\,y_i\,\eta_i - 2\,B_i\,y_i\,\beta_i\,\eta_i - \gamma\,\hbar\,y_i^2\,\eta_i^2\right)\epsilon}{2\,B_i^2} + O[\epsilon]^2\right]$$

$$dS \to \mathbb{E}_{\{i\}\to\{i\}}\Big[-a_i\,\alpha_i - b_i\,\beta_i,\ \frac{-\hbar\,y_i\,\mathcal{A}_i\,\eta_i - \hbar\,B_i\,x_i\,\mathcal{A}_i\,\xi_i + \mathcal{A}_i\,\eta_i\,\xi_i - B_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{\hbar\,B_i},$$
$$1 + \frac{1}{4\,\hbar\,B_i^2}\Big(4\,\gamma\,\hbar^2\,B_i\,y_i\,\mathcal{A}_i\,\eta_i - 4\,\hbar\,B_i\,y_i\,\mathcal{A}_i\,\beta_i\,\eta_i - 2\,\gamma\,\hbar^2\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2 - 4\,\hbar^2\,a_i\,B_i^2\,x_i\,\mathcal{A}_i\,\xi_i - 4\,\hbar\,B_i^2\,x_i\,\mathcal{A}_i\,\beta_i\,\xi_i -$$
$$4\,\gamma\,\hbar\,B_i\,\mathcal{A}_i\,\eta_i\,\xi_i + 4\,\hbar\,a_i\,B_i\,\mathcal{A}_i\,\eta_i\,\xi_i + 4\,\gamma\,\hbar\,B_i^2\,\mathcal{A}_i\,\eta_i\,\xi_i - 4\,\gamma\,\hbar^2\,B_i\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i +$$
$$4\,B_i\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i - 4\,B_i^2\,\mathcal{A}_i\,\beta_i\,\eta_i\,\xi_i + 6\,\gamma\,\hbar\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i - 2\,\gamma\,\hbar\,B_i\,y_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i - 2\,\gamma\,\hbar^2\,B_i^2\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 +$$
$$6\,\gamma\,\hbar\,B_i\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 2\,\gamma\,\hbar\,B_i^2\,x_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i^2 - 3\,\gamma\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 + 4\,\gamma\,B_i\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2 - \gamma\,B_i^2\,\mathcal{A}_i^2\,\eta_i^2\,\xi_i^2\Big)\,\epsilon + O[\epsilon]^2\Big]$$

$$a\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i + a_k\,\alpha_i,\ x_j\,\xi_i + x_k\,\xi_i,\ 1 + \frac{1}{2}\left(-2\,\hbar\,a_j\,x_k\,\xi_i + \gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon + O[\epsilon]^2\right]$$

$$b\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\left[b_j\,\beta_i + b_k\,\beta_i,\ B_k\,y_j\,\eta_i + y_k\,\eta_i,\ 1 + \frac{1}{2}\,\gamma\,\hbar\,B_k\,y_j\,y_k\,\eta_i^2\,\epsilon + O[\epsilon]^2\right]$$

$$d\Delta \to \mathbb{E}_{\{i\}\to\{j,k\}}\Big[a_j\,\alpha_i + a_k\,\alpha_i + b_j\,\beta_i + b_k\,\beta_i,$$
$$y_j\,\eta_i + B_j\,y_k\,\eta_i + x_j\,\xi_i + x_k\,\xi_i,\ 1 + \frac{1}{2}\left(\gamma\,\hbar\,B_j\,y_j\,y_k\,\eta_i^2 - 2\,\hbar\,a_j\,x_k\,\xi_i + \gamma\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon + O[\epsilon]^2\Big]$$

$$C \to \mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \sqrt{B_i} - \frac{1}{2}\left(\hbar\,a_i\,\sqrt{B_i}\right)\epsilon + O[\epsilon]^2\right]$$

$$\overline{C} \to \mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \frac{1}{\sqrt{B_i}} + \frac{\hbar\,a_i\,\epsilon}{2\,\sqrt{B_i}} + O[\epsilon]^2\right]$$

$$Kink \to \mathbb{E}_{\{\}\to\{i\}}\left[\hbar\,a_i\,b_i,\ \hbar\,x_i\,y_i,\ \frac{1}{\sqrt{B_i}} + \frac{\left(2\,\hbar\,a_i - \gamma\,\hbar^3\,x_i^2\,y_i^2\right)\epsilon}{4\,\sqrt{B_i}} + O[\epsilon]^2\right]$$

$$\overline{Kink} \to \mathbb{E}_{\{\}\to\{i\}}\left[-\hbar\,a_i\,b_i,\ -\frac{\hbar\,x_i\,y_i}{B_i},\ \sqrt{B_i} + \frac{\left(-2\,\hbar\,a_i\,B_i^2 - 4\,\hbar^2\,a_i\,B_i\,x_i\,y_i - 3\,\gamma\,\hbar^3\,x_i^2\,y_i^2\right)\epsilon}{4\,B_i^{3/2}} + O[\epsilon]^2\right]$$

$$b2t \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i - \frac{t_i\,\beta_i}{\gamma},\ y_i\,\eta_i + x_i\,\xi_i,\ 1 + \frac{a_i\,\beta_i\,\epsilon}{\gamma} + O[\epsilon]^2\right]$$

$$t2b \to \mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i - \gamma\,b_i\,\tau_i,\ y_i\,\eta_i + x_i\,\xi_i,\ 1 + a_i\,\tau_i\,\epsilon + O[\epsilon]^2\right]$$

Check that on the generators this agrees with our conventions in the handout:

In[∘]:=
```
Timing@
 {{"[a,x]" → ((𝔼_{{}→{1,2}}[0, 0, a_2 x_1] // am_{1,2→1})[3] - (𝔼_{{}→{1,2}}[0, 0, a_1 x_2] // am_{1,2→1})[3]),
    "[b,y]" → ((𝔼_{{}→{1,2}}[0, 0, y_2 b_1] // bm_{1,2→1})[3] - (𝔼_{{}→{1,2}}[0, 0, y_1 b_2] // bm_{1,2→1})[3])} /.
   z_1 → z,
  {"Δ[y]" → Last[𝔼_{{}→{1}}[0, 0, y_1] ~B_1~ bΔ_{1→1,2}],
   "Δ[b]" → Last[𝔼_{{}→{1}}[0, 0, b_1] ~B_1~ bΔ_{1→1,2}],
   "Δ[a]" → Last[𝔼_{{}→{1}}[0, 0, a_1] ~B_1~ aΔ_{1→1,2}],
   "Δ[x]" → Last[𝔼_{{}→{1}}[0, 0, x_1] ~B_1~ aΔ_{1→1,2}]},
  {
    "S(a)" → ((𝔼_{{}→{1}}[0, 0, a_1] ~B_1~ aS_1)[3]),
    "S(x)" → ((𝔼_{{}→{1}}[0, 0, x_1] ~B_1~ aS_1)[3]),
    "S(b)" → ((𝔼_{{}→{1}}[0, 0, b_1] ~B_1~ bS_1)[3]),
    "S(y)" → ((𝔼_{{}→{1}}[0, 0, y_1] ~B_1~ bS_1)[3])
   } /. z_1 → z}
```

Out[∘]=

$$\{0.90625, \{\{[a,x] \to -x\gamma,\ [b,y] \to -y\epsilon + O[\epsilon]^3\},\ \{\Delta[y] \to (B_2 y_1 + y_2) + O[\epsilon]^3,\ \Delta[b] \to (b_1 + b_2) + O[\epsilon]^3,$$
$$\Delta[a] \to (a_1 + a_2) + O[\epsilon]^3,\ \Delta[x] \to (x_1 + x_2) - \hbar a_1 x_2 \epsilon + \frac{1}{2}\hbar^2 a_1^2 x_2 \epsilon^2 + O[\epsilon]^3\},\ \{S(a) \to -a + O[\epsilon]^3,$$
$$S(x) \to -x - a x \hbar \epsilon - \frac{1}{2}(a^2 x \hbar^2)\epsilon^2 + O[\epsilon]^3,\ S(b) \to -b + O[\epsilon]^3,\ S(y) \to -\frac{y}{B} + O[\epsilon]^3\}\}\}$$

Hopf algebra axioms on both sides separately.

Associativity of am and bm:

In[∘]:=
```
Timing@Block[{$k = 3},
  HL /@ {(am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}
 ]
```

Out[∘]= {0.140625, {True, True}}

R and *P* are inverses:

In[∘]:=
```
Timing@Block[{$k = 3}, {R_{i,j}, P_{i,k}, HL[(R_{i,j} // P_{i,k}) ≡ 𝔼_{{k}→{j}}[a_j α_k, x_j ξ_k, 1]]}]
```

Out[∘]=

$$\{0.125, \{\mathbb{E}_{\{\}\to\{i,j\}}[\hbar a_j b_i, \hbar x_j y_i, 1 - \frac{1}{4}(\gamma \hbar^3 x_j^2 y_i^2)\epsilon + \left(\frac{1}{9}\gamma^2\hbar^5 x_j^3 y_i^3 + \frac{1}{32}\gamma^2\hbar^6 x_j^4 y_i^4\right)\epsilon^2 +$$
$$\frac{1}{1152}(24\gamma^3\hbar^5 x_j^2 y_i^2 - 72\gamma^3\hbar^7 x_j^4 y_i^4 - 32\gamma^3\hbar^8 x_j^5 y_i^5 - 3\gamma^3\hbar^9 x_j^6 y_i^6)\epsilon^3 + O[\epsilon]^4],$$
$$\mathbb{E}_{\{i,k\}\to\{\}}[\frac{\alpha_k \beta_i}{\hbar}, \frac{\eta_i \xi_k}{\hbar}, 1 + \frac{\gamma \eta_i^2 \xi_k^2 \epsilon}{4\hbar} + \frac{(36\gamma^2\hbar^2\eta_i^2\xi_k^2 + 40\gamma^2\hbar\eta_i^3\xi_k^3 + 9\gamma^2\eta_i^4\xi_k^4)\epsilon^2}{288\hbar^2} - \frac{1}{1152\hbar^3}$$
$$(-48\gamma^3\hbar^4\eta_i^2\xi_k^2 - 192\gamma^3\hbar^3\eta_i^3\xi_k^3 - 156\gamma^3\hbar^2\eta_i^4\xi_k^4 - 40\gamma^3\hbar\eta_i^5\xi_k^5 - 3\gamma^3\eta_i^6\xi_k^6)\epsilon^3 + O[\epsilon]^4], \text{True}\}\}$$

as and $\overline{\text{aS}}$ are inverses, bs and $\overline{\text{bS}}$ are inverses:

In[∘]:=
```
Timing[HL /@ {(aS̄_1 // aS_1) ≡ 𝔼_{{1}→{1}}[a_1 α_1, x_1 ξ_1, 1], (bS̄_1 // bS_1) ≡ 𝔼_{{1}→{1}}[b_1 β_1, y_1 η_1, 1]}]
```

Out[∘]= {0.359375, {True, True}}

(co)-associativity on both sides

```
In[]:= Timing[
   HL /@ {(aΔ_{1→1,2} // aΔ_{2→2,3}) ≡ (aΔ_{1→1,3} // aΔ_{1→1,2}), (bΔ_{1→1,2} // bΔ_{2→2,3}) ≡ (bΔ_{1→1,3} // bΔ_{1→1,2}),
     (am_{1,2→1} // am_{1,3→1}) ≡ (am_{2,3→2} // am_{1,2→1}), (bm_{1,2→1} // bm_{1,3→1}) ≡ (bm_{2,3→2} // bm_{1,2→1})}]
Out[]= {0.421875, {True, True, True, True}}
```

Δ is an algebra morphism

```
In[]:= Timing[HL /@ {(am_{1,2→1} // aΔ_{1→1,2}) ≡ ((aΔ_{1→1,3} aΔ_{2→2,4}) // (am_{3,4→2} am_{1,2→1})),
     (bm_{1,2→1} // bΔ_{1→1,2}) ≡ ((bΔ_{1→1,3} bΔ_{2→2,4}) // (bm_{3,4→2} bm_{1,2→1}))}]
Out[]= {0.828125, {True, True}}
```

An explicit formula for aS_i

```
In[]:= Timing@Block[{$k = 4}, HL[aS_i ≡ (𝔼_{{i}→{i,j}}[-α_i a_j, -ξ_i x_i,
       Sum[Expand[(ⅇ^{ξ_i x_i} (-ℏ γ ϵ)^k)/(2^k k!) Nest[Expand[x_i^2 ∂_{{x_i,2}} #] &, ⅇ^{-ξ_i ⅇ^{ℏ ϵ a_i} x_i}, k]], {k, 0, $k}]]_{$k} //
     am_{i,j→i})]]
Out[]= {3.15625, True}
```

S is convolution inverse of id

```
In[]:= Timing[HL[# ≡ 𝔼_{{1}→{1}}[0, 0, 1]] & /@ {
     (aΔ_{1→1,2} ~ B_1 ~ aS_1) ~ B_{1,2} ~ am_{1,2→1}, (aΔ_{1→1,2} ~ B_2 ~ aS_2) ~ B_{1,2} ~ am_{1,2→1},
     (bΔ_{1→1,2} ~ B_1 ~ bS_1) ~ B_{1,2} ~ bm_{1,2→1}, (bΔ_{1→1,2} ~ B_2 ~ bS_2) ~ B_{1,2} ~ bm_{1,2→1}}]
Out[]= {0.59375, {True, True, True, True}}
```

But not with the opposite product:

```
In[]:= Timing[Short[# ≡ 𝔼_{{1}→{1}}[0, 0, 1]] & /@ {
     (aΔ_{1→1,2} ~ B_1 ~ aS_1) ~ B_{1,2} ~ am_{2,1→1}, (aΔ_{1→1,2} ~ B_2 ~ aS_2) ~ B_{1,2} ~ am_{2,1→1},
     (bΔ_{1→1,2} ~ B_1 ~ bS_1) ~ B_{1,2} ~ bm_{2,1→1}, (bΔ_{1→1,2} ~ B_2 ~ bS_2) ~ B_{1,2} ~ bm_{2,1→1}}]
```

Out[]= $\left\{0.640625, \left\{\frac{1}{2}\left(-2\gamma\epsilon\hbar x_1 \mathcal{A}_1 \xi_1 + \gamma^2\epsilon^2\hbar^2 x_1 \mathcal{A}_1 \xi_1 - 2\gamma \ll 4 \gg \mathcal{A}_1 \xi_1 + 2\gamma^2\epsilon^2\hbar^2 x_1^2 \mathcal{A}_1^2 \xi_1^2\right) == 0,\right.\right.$
$\frac{1}{2}\left(-2\gamma\epsilon\hbar x_1 \xi_1 - \gamma^2\epsilon^2\hbar^2 x_1 \xi_1 + 2\gamma^2\epsilon^2\hbar^2 x_1^2 \xi_1^2\right) == 0,$
$\left.\left.\frac{1}{2}\left(-2\gamma\epsilon\hbar y_1 \eta_1 - \gamma^2\epsilon^2\hbar^2 y_1 \eta_1 + 2\gamma^2\epsilon^2\hbar^2 y_1^2 \eta_1^2\right) == 0, \frac{-2\gamma\epsilon\hbar B_1 y_1 \eta_1 + \ll 4 \gg}{2 B_1^2} == 0\right\}\right\}$

S is an algebra anti-(co)morphism

```
In[]:= Timing[HL /@ {am_{1,2→1} ~ B_1 ~ aS_1 ≡ (aS_1 aS_2) ~ B_{1,2} ~ am_{2,1→1}, bm_{1,2→1} ~ B_1 ~ bS_1 ≡ (bS_1 bS_2) ~ B_{1,2} ~ bm_{2,1→1},
     aS_1 ~ B_1 ~ aΔ_{1→1,2} ≡ aΔ_{1→2,1} ~ B_{1,2} ~ (aS_1 aS_2), bS_1 ~ B_1 ~ bΔ_{1→1,2} ≡ bΔ_{1→2,1} ~ B_{1,2} ~ (bS_1 bS_2)}]
Out[]= {0.890625, {True, True, True, True}}
```

Pairing axioms

In[∘]:=
```
Timing[HL /@ {(bm1,2→1 𝔼{3}→{3}[α3 a3, ξ3 x3, 1]) ~ B1,3 ~ P1,3 ≡
    (𝔼{1}→{1}[β1 b1, η1 y1, 1] 𝔼{2}→{2}[β2 b2, η2 y2, 1] aΔ3→4,5) ~ B1,4 ~ P1,4 ~ B2,5 ~ P2,5,
   (bΔ1→1,2 𝔼{3}→{3}[α3 a3, ξ3 x3, 1] 𝔼{4}→{4}[α4 a4, ξ4 x4, 1]) ~ B1,3 ~ P1,3 ~ B2,4 ~ P2,4 ≡
    (𝔼{1}→{1}[β1 b1, η1 y1, 1] am3,4→3) ~ B1,3 ~ P1,3 }]
```

Out[∘]= {0.40625, {True, True}}

In[∘]:=
```
Timing[HL /@ {((bS1 𝔼{2}→{2}[α2 a2, ξ2 x2, 1]) // P1,2) ≡ ((𝔼{1}→{1}[β1 b1, η1 y1, 1] aS2) // P1,2),
   (b̄S̄1 𝔼{2}→{2}[α2 a2, ξ2 x2, 1]) ~ B1,2 ~ P1,2 ≡ (𝔼{1}→{1}[β1 b1, η1 y1, 1] āS̄2) ~ B1,2 ~ P1,2}]
```

Out[∘]= {0.265625, {True, True}}

Tests for the double.

Check the double formulas on the generators agree with SL2Portfolio.pdf:

In[∘]:=
```
Timing@{{
     "[a,y]" →
      ((𝔼{}→{1,2}[0, 0, y2 a1] ~ B1,2 ~ dm1,2→1)[3] - (𝔼{}→{1,2}[0, 0, y1 a2] ~ B1,2 ~ dm1,2→1)[3]),
     "[b,x]"  → ((𝔼{}→{1,2}[0, 0, x2 b1] ~ B1,2 ~ dm1,2→1)[3] -
        (𝔼{}→{1,2}[0, 0, x1 b2] ~ B1,2 ~ dm1,2→1)[3]),
     "xy-qyx" → ((𝔼{}→{1,2}[0, 0, x1 y2] ~ B1,2 ~ dm1,2→1)[3] -
        (1 + ϵ) (𝔼{}→{1,2}[0, 0, y1 x2] ~ B1,2 ~ dm1,2→1)[3])
    } /. {z_1 → z} // Expand // Factor,
  {
    "Δ(a)" → ((𝔼{}→{1}[0, 0, a1] ~ B1 ~ dΔ1→1,2)[3]),
    "Δ(x)" → ((𝔼{}→{1}[0, 0, x1] ~ B1 ~ dΔ1→1,2)[3]),
    "Δ(b)" → ((𝔼{}→{1}[0, 0, b1] ~ B1 ~ dΔ1→1,2)[3]),
    "Δ(y)" → ((𝔼{}→{1}[0, 0, y1] ~ B1 ~ dΔ1→1,2)[3])
   } // Simplify,
  {
     "S(a)" → ((𝔼{}→{1}[0, 0, a1] ~ B1 ~ dS1)[3]),
     "S(x)" → ((𝔼{}→{1}[0, 0, x1] ~ B1 ~ dS1)[3]),
     "S(b)" → ((𝔼{}→{1}[0, 0, b1] ~ B1 ~ dS1)[3]),
     "S(y)" → ((𝔼{}→{1}[0, 0, y1] ~ B1 ~ dS1)[3])
    } /. {z_1 → z} // Simplify
 }
```

Out[∘]=
$$\Big\{7.4375,\ \Big\{\Big\{[a,y]\to -y\,\gamma + O[\epsilon]^3,\ [b,x]\to x\,\epsilon + O[\epsilon]^3,$$
$$xy\text{-}qyx \to \left(-x\,y + \frac{1-B+x\,y\,\hbar}{\hbar}\right) + (a\,B - x\,y + x\,y\,\gamma\,\hbar)\,\epsilon + \frac{1}{2}\left(-a^2\,B\,\hbar + x\,y\,\gamma^2\,\hbar^2\right)\epsilon^2 + O[\epsilon]^3\Big\},$$
$$\Big\{\Delta(a)\to (a_1+a_2) + O[\epsilon]^3,\ \Delta(x)\to (x_1+x_2) - \hbar\,a_1\,x_2\,\epsilon + \frac{1}{2}\hbar^2\,a_1^2\,x_2\,\epsilon^2 + O[\epsilon]^3,$$
$$\Delta(b)\to (b_1+b_2) + O[\epsilon]^3,\ \Delta(y)\to (y_1 + B_1\,y_2) + O[\epsilon]^3\Big\},$$
$$\Big\{S(a)\to -a + O[\epsilon]^3,\ S(x)\to -x - a\,x\,\hbar\,\epsilon - \frac{1}{2}\left(a^2\,x\,\hbar^2\right)\epsilon^2 + O[\epsilon]^3,$$
$$S(b)\to -b + O[\epsilon]^3,\ S(y)\to -\frac{y}{B} + \frac{y\,\gamma\,\hbar\,\epsilon}{B} - \frac{\left(y\,\gamma^2\,\hbar^2\right)\epsilon^2}{2\,B} + O[\epsilon]^3\Big\}\Big\}\Big\}$$

(co)-associativity

In[∘]:= Timing[
HL /@ {(dΔ$_{1\to1,2}$ // dΔ$_{2\to2,3}$) ≡ (dΔ$_{1\to1,3}$ // dΔ$_{1\to1,2}$), (dm$_{1,2\to1}$ // dm$_{1,3\to1}$) ≡ (dm$_{2,3\to2}$ // dm$_{1,2\to1}$)}]

Out[∘]= {7.3125, {True, True}}

Δ is an algebra morphism

In[∘]:= Timing@HL[dm$_{1,2\to1}$ ~ B$_1$ ~ dΔ$_{1\to1,2}$ ≡ (dΔ$_{1\to1,3}$ dΔ$_{2\to2,4}$) ~ B$_{1,2,3,4}$ ~ (dm$_{3,4\to2}$ dm$_{1,2\to1}$)]

Out[∘]= {8.14063, True}

S_2 inverts R, but not S_1:

In[∘]:= Timing@{R$_{1,2}$ ~ B$_1$ ~ dS$_1$ ≡ $\overline{R}_{1,2}$, HL[R$_{1,2}$ ~ B$_2$ ~ dS$_2$ ≡ $\overline{R}_{1,2}$]}

Out[∘]= $\left\{0.71875, \left\{\frac{1}{4 B_1^3}\left(4 \gamma \epsilon \hbar^2 B_1^2 x_2 y_1 - 2 \gamma^2 \epsilon^2 \hbar^3 B_1^2 x_2 y_1 + 4 \gamma \epsilon^2 \hbar^3 a_2 B_1^2 x_2 y_1 + 8 \gamma^2 \epsilon^2 \hbar^4 B_1 x_2^2 y_1^2 - 4 \gamma \epsilon^2 \hbar^4 a_2 B_1 x_2^2 y_1^2 - 3 \gamma^2 \epsilon^2 \hbar^5 x_2^3 y_1^3\right) == 0, \text{True}\right\}\right\}$

S is convolution inverse of id

In[∘]:= Timing[HL[# ≡ 𝔼$_{\{1\}\to\{1\}}$[0, 0, 1]] & /@
{(dΔ$_{1\to1,2}$ ~ B$_1$ ~ dS$_1$) ~ B$_{1,2}$ ~ dm$_{1,2\to1}$, (dΔ$_{1\to1,2}$ ~ B$_2$ ~ dS$_2$) // dm$_{1,2\to1}$}]

Out[∘]= {9.54688, {True, True}}

S is a (co)-algebra anti-morphism

In[∘]:= Timing[HL /@
Expand /@ {dm$_{1,2\to1}$ ~ B$_1$ ~ dS$_1$ ≡ (dS$_1$ dS$_2$) ~ B$_{1,2}$ ~ dm$_{2,1\to1}$, dS$_1$ ~ B$_1$ ~ dΔ$_{1\to1,2}$ ≡ dΔ$_{1\to2,1}$ ~ B$_{1,2}$ ~ (dS$_1$ dS$_2$)}]

Out[∘]= {18.4219, {True, True}}

Quasi-triangular axiom 1:

In[∘]:= Timing@HL[R$_{1,2}$ ~ B$_1$ ~ dΔ$_{1->1,3}$ ≡ (R$_{1,4}$ R$_{3,2}$) ~ B$_{2,4}$ ~ dm$_{2,4\to2}$]

Out[∘]= {0.75, True}

Quasi-triangular axiom 2:

In[∘]:= Timing@HL[((dΔ$_{1->1,2}$ R$_{3,4}$) ~ B$_{1,2,3,4}$ ~ (dm$_{1,3\to1}$ dm$_{2,4\to2}$)) ≡ ((dΔ$_{1\to2,1}$ R$_{3,4}$) ~ B$_{1,2,3,4}$ ~ (dm$_{3,1\to1}$ dm$_{4,2\to2}$))]

Out[∘]= {7.54688, True}

The Drinfel'd element inverse property, $(u_1\, \overline{u}_2)$~$B_{1,2}$~dm$_{1,2\to1}$ ≡ 𝔼[0, 0, 1]:

In[∘]:= Timing@HL[((R$_{1,2}$ ~ B$_1$ ~ dS$_1$ ~ B$_{1,2}$ ~ dm$_{2,1\to i}$) (R$_{1,2}$ ~ B$_2$ ~ dS$_2$ ~ B$_2$ ~ dS$_2$ ~ B$_{1,2}$ ~ dm$_{2,1\to j}$)) ~ B$_{i,j}$ ~ dm$_{i,j\to i}$ ≡
𝔼$_{\{\}\to\{i\}}$[0, 0, 1]]

Out[∘]= {3.89063, True}

The ribbon element v satisfies $v^2 = S(u)\, u$. The spinner C=uv^{-1}. It is convenient to compute $z = S(u)\, u^{-1}$ which is something easy.

In[∘]:=
$$\texttt{Timing@Block}\Big[\{\$k = 2\},\ \Big(\Big(\big(R_{1,2}\sim B_1\sim dS_1\sim B_{1,2}\sim dm_{2,1\to i}\big)\sim B_i\sim dS_i\Big)\ \big(R_{1,2}\sim B_2\sim dS_2\sim B_2\sim dS_2\sim B_{1,2}\sim dm_{2,1\to j}\big)\Big)\sim B_{i,j}\sim dm_{i,j\to i}\Big]$$

Out[∘]=
$$\Big\{4.125,\ \mathbb{E}_{\{\}\to\{i\}}\Big[0,\ 0,\ \frac{1}{B_i}+\frac{\hbar\, a_i\, \epsilon}{B_i}+\frac{\hbar^2\, a_i^2\, \epsilon^2}{2\, B_i}+O[\epsilon]^3\Big]\Big\}$$

In[∘]:=
$$\texttt{Timing@Block}\Big[\{\$k = 2\},\ \texttt{HL /@}\Big\{\big(C_i\,\overline{C}_j\big)\sim B_{i,j}\sim dm_{i,j\to i} \equiv \mathbb{E}_{\{\}\to\{i\}}[0,0,1],\ \big(\overline{C}_i\,\overline{C}_j\big)\sim B_{i,j}\sim dm_{i,j\to i} \equiv \Big(\Big(\big(R_{1,2}\sim B_1\sim dS_1\sim B_{1,2}\sim dm_{2,1\to i}\big)\sim B_i\sim dS_i\Big)\ \big(R_{1,2}\sim B_2\sim dS_2\sim B_2\sim dS_2\sim B_{1,2}\sim dm_{2,1\to j}\big)\Big)\sim B_{i,j}\sim dm_{i,j\to i}\Big\}\Big]$$

Out[∘]= {4.90625, {True, True}}

Reidemeister 2:

In[∘]:=
$$\texttt{Timing}\Big[\texttt{HL}[\#\equiv \mathbb{E}_{\{\}\to\{1,2\}}[0,0,1]]\ \&\ \texttt{/@}\ \Big\{\big(\overline{R}_{1,2}\,R_{3,4}\big)\sim B_{1,2,3,4}\sim\big(dm_{1,3\to 1}\,dm_{2,4\to 2}\big),\ \big(R_{1,2}\,\overline{R}_{3,4}\big)\sim B_{1,2,3,4}\sim\big(dm_{1,3\to 1}\,dm_{2,4\to 2}\big)\Big\}\Big]$$

Out[∘]= {5.51563, {True, True}}

Cyclic Reidemeister 2:

In[∘]:=
$$\texttt{Timing@HL}\Big[\big(R_{1,4}\,\overline{R}_{5,2}\,\overline{C}_3\big)\sim B_{2,4}\sim dm_{2,4\to 2}\sim B_{1,3}\sim dm_{1,3\to 1}\sim B_{1,5}\sim dm_{1,5\to 1}\equiv \overline{C}_1\,\mathbb{E}_{\{\}\to\{2\}}[0,0,1]\Big]$$

Out[∘]= {2.14063, True}

Reidemeister 3:

In[∘]:=
$$\texttt{Timing@HL}\Big[\big((R_{1,2}\,R_{4,3}\,R_{5,6})\sim B_{1,4}\sim dm_{1,4\to 1}\sim B_{2,5}\sim dm_{2,5\to 2}\sim B_{3,6}\sim dm_{3,6\to 3}\big)\equiv \big((R_{1,6}\,R_{2,3}\,R_{4,5})\sim B_{1,4}\sim dm_{1,4\to 1}\sim B_{2,5}\sim dm_{2,5\to 2}\sim B_{3,6}\sim dm_{3,6\to 3}\big)\Big]$$

Out[∘]= {6.10938, True}

Relations between the four kinks:

In[∘]:=
$$\texttt{Timing}\Big[\texttt{HL /@}\Big\{\texttt{Kink}_i \equiv \big(R_{3,1}\,C_2\big)\sim B_{1,2}\sim dm_{1,2\to 1}\sim B_{1,3}\sim dm_{1,3\to i},\ \overline{\texttt{Kink}}_j \equiv \big(\overline{R}_{3,1}\,\overline{C}_2\big)\sim B_{1,2}\sim dm_{1,2\to 1}\sim B_{1,3}\sim dm_{1,3\to j},\ \big(\texttt{Kink}_i\,\overline{\texttt{Kink}}_j\big)\sim B_{i,j}\sim dm_{i,j\to 1}\equiv \mathbb{E}_{\{\}\to\{1\}}[0,0,1]\Big\}\Big]$$

Out[∘]= {6.75, {True, True, True}}

Trefoil

The Trefoil

Trefoil

In[∘]:=
```
$k = 2; Z = kR_{1,5} kR_{6,2} kR_{3,7} \overline{kC_4} \overline{kKink_8} \overline{kKink_9} \overline{kKink_{10}};
Do[Z = Z ~ B_{1,r} ~ km_{1,r→1}, {r, 2, 10}];
Simplify /@ Z /. ν_1 :→ ν
```

Trefoil

Out[∘]=
$$\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{T}{1-T+T^2}+\left(T\hbar\left(2a\left(-1+T-T^3+T^4\right)+T\left(-1+2T-3T^2+2T^3\right)\gamma-2\left(1+T^3\right)xy\gamma\hbar\right)\epsilon\right)\Big/\left(1-T+T^2\right)^3+\frac{1}{2\left(1-T+T^2\right)^5}T\hbar^2\Big(4a^2\left(1-T+T^2\right)^2\left(1+T-6T^2+T^3+T^4\right)+4a\left(1-T+T^2\right)\gamma\left(T\left(2-5T+8T^2-7T^3-2T^4+2T^5\right)-2\left(-1-2T+5T^2-4T^3+T^4+2T^5\right)xy\hbar\right)+\gamma^2\Big(T\left(1-2T+4T^2-2T^3+6T^5-11T^6+4T^7\right)+4\left(-1+2T+T^3+T^4+2T^6-T^7\right)xy\hbar+6\left(1-T+T^2\right)^2\left(1+3T+T^2\right)x^2y^2\hbar^2\Big)\Big)\epsilon^2+O[\epsilon]^3\Big]$$