Some Feynman Diagrams in Algebra

With Roland van der Veen

Abstract. I will explain how the computation of compositions of maps of a certain natural class, from one polynomial ring into another, naturally leads to a certain composition operation of quadratics and to Feynman diagrams. Possibly in a later seminar I will explain how this technology can be used to construct a tangle invariant whose need was explained in a talk I gave in the topology seminar on Sep 16 (ωεβ/top).

Secret Slide (must not be shown). Il y a beaucoup de $Z\colon \{\text{Noeuds}\} \to \big(\mathcal{U}(\mathfrak{g}) \text{ ou } \mathcal{U}_q(\mathfrak{g})\big) \cong \mathcal{S}(\mathfrak{g}) \cong \mathbb{Q}[z_1, z_2, \ldots]$.

Conventions. 1. For a set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$. 2. Everything converges!

The Generating Series $\mathcal{D}$: $\operatorname{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B]) \to \mathbb{Q}[\![\zeta_A, z_B]\!]$.

Claim. $F \in \operatorname{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B]) \xrightarrow[\mathcal{D}]{\sim} \mathbb{Q}[z_B][\![\zeta_A]\!] \ni \mathcal{F}$ via

$$\mathcal{D}(F) := \sum_{n\in\mathbb{N}^A} \frac{\zeta_A^n}{n!} F(z_A^n) = F\left(\mathbb{e}^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{F},$$

$$\mathcal{D}^{-1}(\mathcal{F})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{F}\right)_{\zeta_a=0} \quad \text{for } p \in \mathbb{Q}[z_A].$$

Claim. If $F \in \operatorname{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$, $G \in \operatorname{Hom}(\mathbb{Q}[z_B] \to \mathbb{Q}[z_C])$, $\mathcal{F} = \mathcal{D}(F)$, and $\mathcal{G} = \mathcal{D}(G)$, then

$$\mathcal{D}(F/\!\!/G) = \left(\mathcal{F}|_{z_b\to\partial_{\zeta_b}}\mathcal{G}\right)_{\zeta_b=0}.$$

Basic Examples. 1. $\mathcal{D}(id\colon \mathbb{Q}[y,a,x] \to \mathbb{Q}[y,a,x]) = \mathbb{e}^{\eta y+\alpha a+\xi x}$.

2. The standard commutative product m_k^{ij} of polynomials is given by $z_i, z_j \to z_k$. Hence $\mathcal{D}(m_k^{ij}) = m_k^{ij}(\mathbb{e}^{\zeta_i z_i+\zeta_j z_j}) = \mathbb{e}^{(\zeta_i+\zeta_j)z_k}$.

$$\begin{array}{ccc} \mathbb{Q}[z]_i \otimes \mathbb{Q}[z]_j & \xrightarrow{m_k^{ij}} & \mathbb{Q}[z]_k \\ \| & & \| \\ \mathbb{Q}[z_i, z_j] & \xrightarrow{m_k^{ij}} & \mathbb{Q}[z_k] \end{array}$$

3. The standard co-commutative coproduct Δ^i_{jk} of polynomials is given by $z_i \to z_j + z_k$. Hence $\mathcal{D}(\Delta^i_{jk}) = \Delta^i_{jk}(\mathbb{e}^{\zeta_i z_i}) = \mathbb{e}^{\zeta_i(z_j+z_k)}$.

$$\begin{array}{ccc} \mathbb{Q}[z]_i & \xrightarrow{\Delta^i_{jk}} & \mathbb{Q}[z]_j \otimes \mathbb{Q}[z]_k \\ \| & & \| \\ \mathbb{Q}[z_i] & \xrightarrow{\Delta^i_{jk}} & \mathbb{Q}[z_j, z_k] \end{array}$$

Heisenberg Algebras. Let $\mathbb{H} = \langle x, y\rangle/[x,y] = \hbar$ (with $\hbar$ a scalar), let $\mathbb{O}_i\colon \mathbb{Q}[x_i, y_i] \to \mathbb{H}_i$ is the "x before y" PBW ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[x_i, y_i, x_j, y_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathbb{H}_i \otimes \mathbb{H}_j \xrightarrow{m_k^{ij}} \mathbb{H}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[x_k, y_k].$$

Then $\mathcal{D}(hm_k^{ij}) = \mathbb{e}^{\Lambda_\hbar}$, where $\Lambda_\hbar = -\hbar\eta_i\xi_j + (\xi_i+\xi_j)x_k + (\eta_i+\eta_j)y_k$.

Proof 1. Recall the "Weyl form of the CCR" $\mathbb{e}^{\eta y}\mathbb{e}^{\xi x} = \mathbb{e}^{-\hbar\eta\xi}\mathbb{e}^{\xi x}\mathbb{e}^{\eta y}$, and compute

$$\begin{aligned} \mathcal{D}(hm_k^{ij}) &= \mathbb{e}^{\xi_i x_i+\eta_i y_i+\xi_j x_j+\eta_j y_j} /\!\!/ \mathbb{O}_i \otimes \mathbb{O}_j /\!\!/ m_k^{ij} /\!\!/ \mathbb{O}_k^{-1} \\ &= \mathbb{e}^{\xi_i x_i}\mathbb{e}^{\eta_i y_i}\mathbb{e}^{\xi_j x_j}\mathbb{e}^{\eta_j y_j} /\!\!/ m_k^{ij} /\!\!/ \mathbb{O}_k^{-1} = \mathbb{e}^{\xi_i x_k}\mathbb{e}^{\eta_i y_k}\mathbb{e}^{\xi_j x_k}\mathbb{e}^{\eta_j y_k} /\!\!/ \mathbb{O}_k^{-1} \\ &= \mathbb{e}^{-\hbar\eta_i\xi_j}\mathbb{e}^{(\xi_i+\xi_j)x_k}\mathbb{e}^{(\eta_i+\eta_j)y_k} /\!\!/ \mathbb{O}_k^{-1} = \mathbb{e}^{\Lambda_\hbar}. \end{aligned}$$

Proof 2. We compute in a faithful 3D representation ρ of $\mathbb{H}$: (ωεβ/hm)

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{ρx = (0 1 0; 0 0 0; 0 0 0), ρy = (0 0 0; 0 0 ℏ; 0 0 0), ρc = (0 0 1; 0 0 0; 0 0 0)};
{ρx.ρy - ρy.ρx == ℏ ρc, ρx.ρc == ρc.ρx, ρy.ρc == ρc.ρy}
{True, True, True}
Λ = -ℏ ηi ξj ck + (ξi + ξj) xk + (ηi + ηj) yk;
Simplify@With[{𝔼 = MatrixExp},
  𝔼[ξi ρx].𝔼[ηi ρy].𝔼[ξj ρx].𝔼[ηj ρy] ==
   𝔼[∂xk Λ ρx].𝔼[∂yk Λ ρy].𝔼[∂ck Λ ρc]]
True
```

A real DoPeGDO Example (DoPeGDO:=Docile Perturbed Gaussian Differential Operators). Let $sl_{2+}^\epsilon := L\langle y, b, a, x\rangle$ subject to $[a,x] = x$, $[b,y] = -\epsilon y$, $[a,b] = 0$, $[a,y] = -y$, $[b,x] = \epsilon x$, and $[x,y] = \epsilon a + b$. So $t := \epsilon a - b$ is central and if $\exists\epsilon^{-1}$, $sl_{2+}^\epsilon/\langle t\rangle \cong sl_2$. Let $CU := \mathcal{U}(sl_{2+}^\epsilon)$, and let cm_k^{ij} be the composition below, where $\mathbb{O}_i\colon \mathbb{Q}[y_i, b_i, a_i, x_i] \to CU_i$ be the PBW ordering map in the order $ybax$:

$$\begin{array}{ccc} CU_i \otimes CU_j & \xrightarrow{m_k^{ij}} & CU_k \\ \uparrow \mathbb{O}_{i,j} & & \uparrow \mathbb{O}_k \\ \mathbb{Q}[y_i, b_i, a_i, x_i, y_j, b_j, a_j, x_j] & \xrightarrow{cm_k^{ij}} & \mathbb{Q}[y_k, b_k, a_k, x_k] \end{array}$$

Claim. Let

$$\Lambda = \left(\eta_i + \frac{e^{-\alpha_i-\epsilon\beta_i}\eta_j}{1+\epsilon\eta_j\xi_i}\right)y_k + \left(\beta_i + \beta_j + \frac{\log\left(1+\epsilon\eta_j\xi_i\right)}{\epsilon}\right)b_k +$$
$$\left(\alpha_i + \alpha_j + \log\left(1+\epsilon\eta_j\xi_i\right)\right)a_k + \left(\frac{e^{-\alpha_j-\epsilon\beta_j}\xi_i}{1+\epsilon\eta_j\xi_i} + \xi_j\right)x_k$$

Then $\mathbb{e}^{\eta_i y_i+\beta_i b_i+\alpha_i a_i+\xi_i x_i+\eta_j y_j+\beta_j b_j+\alpha_j a_j+\xi_j x_j} /\!\!/ \mathbb{O}_{i,j} /\!\!/ cm_k^{ij} = \mathbb{e}^\Lambda /\!\!/ \mathbb{O}_k$, and hence $\mathcal{D}(cm_k^{ij}) = \mathbb{e}^\Lambda$.

Proof. We compute in a faithful 2D representation ρ of CU: (ωεβ/cm)

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{ρy = (0 0; ϵ 0), ρb = (0 0; 0 -ϵ), ρa = (1 0; 0 0), ρx = (0 1; 0 0)};
{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy,
 ρb.ρy - ρy.ρb == -ϵ ρy, ρb.ρx - ρx.ρb == ϵ ρx,
 ρx.ρy - ρy.ρx == ρb + ϵ ρa}
{True, True, True, True, True}
Simplify@With[{𝔼 = MatrixExp},
  𝔼[ηi ρy].𝔼[βi ρb].𝔼[αi ρa].𝔼[ξi ρx].𝔼[ηj ρy].𝔼[βj ρb].
   𝔼[αj ρa].𝔼[ξj ρx] ==
  𝔼[∂yk Λ ρy].𝔼[∂bk Λ ρb].𝔼[∂ak Λ ρa].𝔼[∂xk Λ ρx]]
True
Series[Λ, {ϵ, 0, 2}]
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$$\begin{aligned} &\left(a_k(\alpha_i+\alpha_j) + y_k(\eta_i + \mathbb{e}^{-\alpha_i}\eta_j) + b_k(\beta_i+\beta_j+\eta_j\xi_i) + x_k(\mathbb{e}^{-\alpha_j}\xi_i+\xi_j)\right) + \\ &\left(a_k\eta_j\xi_i - \tfrac{1}{2}b_k\eta_j^2\xi_i^2 - \mathbb{e}^{-\alpha_i}y_k\eta_j(\beta_i+\eta_j\xi_i) - \mathbb{e}^{-\alpha_j}x_k\xi_i(\beta_j+\eta_j\xi_i)\right)\epsilon + \\ &\left(-\tfrac{1}{2}a_k\eta_j^2\xi_i^2 + \tfrac{1}{3}b_k\eta_j^3\xi_i^3 + \tfrac{1}{2}\mathbb{e}^{-\alpha_i}y_k\eta_j(\beta_i^2+2\beta_i\eta_j\xi_i+2\eta_j^2\xi_i^2) + \tfrac{1}{2}\mathbb{e}^{-\alpha_j}x_k\xi_i(\beta_j^2+2\beta_j\eta_j\xi_i+2\eta_j^2\xi_i^2)\right)\epsilon^2 + O[\epsilon]^3 \end{aligned}$$

Note 1. If the lower half of the alphabet (a, b, α, β) is regarded as constants, then $\Lambda = C + Q + \sum_{k\geq1}\epsilon^k P^{(k)}$ is a docile perturbed Gaussian relative to the upper half of the alphabet (x, y, ξ, η): C is a scalar, Q is a quadratic, and $\deg P^{(k)} \leq 2k+2$.

Note 2. $\operatorname{wt}(x, y, \xi, \eta, a, b, \alpha, \beta, \epsilon) = (1, 1, 1, 1, 2, 0, 0, 2, -2)$.

Quadratic Casimirs. If $t \in \mathfrak{g}\otimes\mathfrak{g}$ is the quadratic Casimir of a semi-simple Lie algebra $\mathfrak{g}$, then $\mathbb{e}^t$, regarded by PBW as an element of $\mathcal{S}^{\otimes2} = \operatorname{Hom}\left(\mathcal{S}(\mathfrak{g})^{\otimes0} \to \mathcal{S}(\mathfrak{g})^{\otimes2}\right)$, has a latin-latin dominant Gaussian factor. Likewise for R-matrices.

DoPeGDO := The category with objects finite sets[†1] and

$$\operatorname{mor}(A \to B) = \{\mathcal{F} = \omega\exp(Q+P)\} \subset \mathbb{Q}[\![\zeta_A, z_B, \epsilon]\!],$$

where: • ω is a scalar.[†2] • Q is a "small" ϵ-free quadratic in $\zeta_A \cup z_B$.[†3] • P is a "docile perturbation": $P = \sum_{k\geq1}\epsilon^k P^{(k)}$, where $\deg P^{(k)} \leq 2k+2$.[†4] • Compositions:[†6]

$$\mathcal{F}/\!\!/\mathcal{G} = \mathcal{G}\circ\mathcal{F} := \left(\mathcal{G}|_{\zeta_i\to\partial_{z_i}}\mathcal{F}\right)_{z_i=0} = \left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

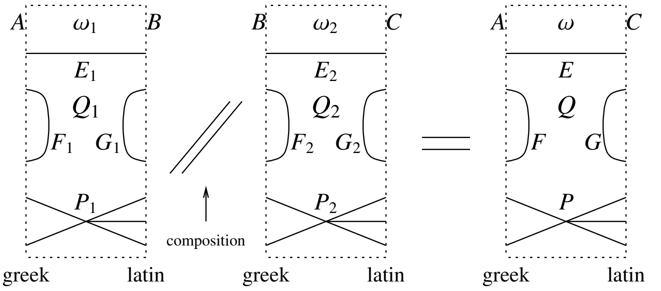
So What? • If V is a representation, then $V^{\otimes n}$ explodes as a function of n, while in **DoPeGDO** and up to a fixed power of ϵ, the ranks of $\mathrm{mor}(A \to B)$ grow polynomially as a function of $|A|$ and $|B|$.

• Approximating sl_{2+}^{ϵ} retains more of its structure then representing it!

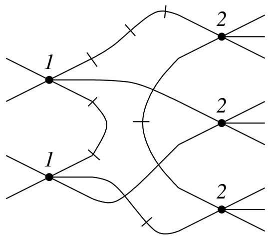
Compositions (1). In $\mathrm{mor}(A \to B)$,

$$Q = \sum_{i\in A, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j,$$

and so



where • $E = E_1(I - F_2G_1)^{-1}E_2$.
• $F = F_1 + E_1F_2(I - G_1F_2)^{-1}E_1^T$.
• $G = G_2 + E_2^T G_1(I - F_2G_1)^{-1}E_2$.
• $\omega = \omega_1\omega_2 \det(I - F_2G_1)^{-1}$.
• P is computed as the solution of a messy PDE or using "connected Feynman diagrams" (yet we're still in algebra!).



DoPeGDO Footnotes. Each variable has a "weight"$\in \{0,1,2\}$, and always $\mathrm{wt}\, z_i + \mathrm{wt}\, \zeta_i = 2$.

†1. Really, "weight-graded finite sets" $A = A_0 \sqcup A_1 \sqcup A_2$.

†2. Really, a power series in the weight-0 variables†5.

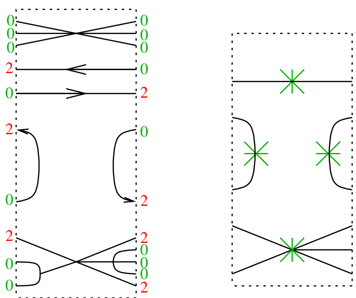
†3. The weight of Q must be 2, so it decomposes as $Q = Q_{20} + Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series†5.

†4. Setting $\mathrm{wt}\,\epsilon = -2$, the weight of P is ≤ 2 (so the powers of the weight-0 variables are not constrained)†5.

†5. In the knot-theoretic case, all weight-0 power series are rational functions of bounded degree in the exponentials of the weight-0 variables.

†6. There's also an obvious product
$\mathrm{mor}(A_1 \to B_1) \times \mathrm{mor}(A_2 \to B_2) \to \mathrm{mor}(A_1 \sqcup A_2 \to B_1 \sqcup B_2)$.

Full DoPeGDO. Compute compositions in two phases:
• A 2-0 phase over $\mathbb{Q}$, in which the weight-1 variables are spectators.
• A 1-1 phase over the ring of power series in the weight-0 variables, in which the weight-2 variables are spectators.



Questions. • Are there QFT precedents for "two-step Gaussian integration"?

• In QFT, one saves even more by considering "one-particle-irreducible" diagrams and "effective actions". Does this mean anything here?

• Understanding $\mathrm{Hom}(\mathbb{Q}[z_A] \to \mathbb{Q}[z_B])$ seems like a good cause. Can you find other applications for the technology here?

Compositions (2). Recall that with all indices i running in some set B,

$$\mathcal{F}/\!/\mathcal{G} = \left(\mathcal{F}|_{z_i \to \partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0} \overset{(1)}{=} \mathbb{e}^{\sum \partial_{z_i}\partial_{\zeta_i}}(\mathcal{F}\mathcal{G})\Big|_{z_i=\zeta_i=0},$$

(1) Strictly speaking, true only when $B \cap (A \cup C) = \emptyset$.

so in general we wish to understand

$$[F : \mathcal{E}]_B := \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad \text{and} \quad \langle F : \mathcal{E}\rangle_B := [F : \mathcal{E}]_B|_{z_B \to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E}\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1 - GF)^{-1/2}\left\langle F(1 - GF)^{-1} : \mathcal{E}\right\rangle_B.$$

The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F : \mathcal{E}\mathbb{e}^{\sum_{i\in B} y_iz_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}y_iy_j}\left\langle F : \mathcal{E}|_{z_B \to z_B + Fy_B}\right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$