The Turbo-Gassner Representation

Pensieve header: The turbo Gassner representation for the June 6 PolyPoly meeting. Based on pensieve://2016-06/.

```
Kδ /: Kδ_is__ := KroneckerDelta[1, Length[Union[{is}]]];
```

The Burau Representation

```
B_i_,j_[ξ_] := ξ /. v_j :> (1 - t) v_i + t v_j

Column@{R3l = {v1, v2, v3} // B1,2 // B1,3 // B2,3,
   R3r = {v1, v2, v3} // B2,3 // B1,3 // B1,2 ,
   R3l - R3r // Expand}
```

```
{v1, (1 - t) v1 + t v2, (1 - t) v1 + t ((1 - t) v2 + t v3)}
{v1, (1 - t) v1 + t v2, (1 - t) ((1 - t) v1 + t v2) + t ((1 - t) v1 + t v3)}
{0, 0, 0}
```

The Gassner Representation

```
G_i_,j_[ξ_] := ξ /. v_j :> (1 - t_i) v_i + t_i v_j

Column@{R3l = {v1, v2, v3} // G1,2 // G1,3 // G2,3,
   R3r = {v1, v2, v3} // G2,3 // G1,3 // G1,2 ,
   R3l - R3r // Expand}
```

```
{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 ((1 - t2) v2 + t2 v3)}
{v1, (1 - t1) v1 + t1 v2, (1 - t2) ((1 - t1) v1 + t1 v2) + t2 ((1 - t1) v1 + t1 v3)}
{0, 0, 0}
```

```
Column@{OCl = {v1, v2, v3} // G1,2 // G1,3,
   OCr = {v1, v2, v3} // G1,3 // G1,2 ,
   OCl - OCr // Expand}
```

```
{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 v3}
{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 v3}
{0, 0, 0}
```

```
Column@{UCl = {v1, v2, v3} // G1,3 // G2,3,
   UCr = {v1, v2, v3} // G2,3 // G1,3 ,
   UCl - UCr // Expand}
```

```
{v1, v2, (1 - t1) v1 + t1 ((1 - t2) v2 + t2 v3)}
{v1, v2, (1 - t2) v2 + t2 ((1 - t1) v1 + t1 v3)}
{0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}
```

The Gassner-Plus Representation

```
GP_{i_,j_}[ℱ_] := Expand[ℱ /. {u_j :> (1 - t_i) u_i + t_i u_j,
      f_ . v_j :> f (1 - t_i) v_i + f t_i v_j +
        (t_i - 1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) u_i + f t_i u_i }];

GPchecks = {f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_2, f[t_1, t_2, t_3] v_3, u_1, u_2, u_3};

R3l = GPchecks // GP_{1,2} // GP_{1,3} // GP_{2,3}
```

$$\{f[t_1, t_2, t_3]\, v_1,\ f[t_1, t_2, t_3]\, t_1\, u_1 + f[t_1, t_2, t_3]\, v_1 - f[t_1, t_2, t_3]\, t_1\, v_1 + f[t_1, t_2, t_3]\, t_1\, v_2 + t_2\, u_1\, f^{(0,1,0)}[t_1, t_2, t_3] - t_1\, t_2\, u_1\, f^{(0,1,0)}[t_1, t_2, t_3] - t_1\, u_1\, f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2\, u_1\, f^{(1,0,0)}[t_1, t_2, t_3],\ f[t_1, t_2, t_3]\, t_1\, u_1 + f[t_1, t_2, t_3]\, t_1\, t_2\, u_2 + f[t_1, t_2, t_3]\, v_1 - f[t_1, t_2, t_3]\, t_1\, v_1 + f[t_1, t_2, t_3]\, t_1\, v_2 - f[t_1, t_2, t_3]\, t_1\, t_2\, v_2 + f[t_1, t_2, t_3]\, t_1\, t_2\, v_3 + t_3\, u_1\, f^{(0,0,1)}[t_1, t_2, t_3] - t_1\, t_3\, u_1\, f^{(0,0,1)}[t_1, t_2, t_3] + t_1\, t_3\, u_2\, f^{(0,0,1)}[t_1, t_2, t_3] - t_1\, t_2\, t_3\, u_2\, f^{(0,0,1)}[t_1, t_2, t_3] - t_1\, t_2\, u_2\, f^{(0,1,0)}[t_1, t_2, t_3] + t_1\, t_2^2\, u_2\, f^{(0,1,0)}[t_1, t_2, t_3] - t_1\, u_1\, f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2\, u_1\, f^{(1,0,0)}[t_1, t_2, t_3],\ u_1,\ u_1 - t_1\, u_1 + t_1\, u_2,\ u_1 - t_1\, u_1 + t_1\, u_2 - t_1\, t_2\, u_2 + t_1\, t_2\, u_3\}$$

```
R3r = GPchecks // GP_{2,3} // GP_{1,3} // GP_{1,2}; R3l - R3r
```

$$\{0, 0, 0, 0, 0, 0\}$$

```
OCl = GPchecks // GP_{1,2} // GP_{1,3}
```

$$\{f[t_1, t_2, t_3]\, v_1,\ f[t_1, t_2, t_3]\, t_1\, u_1 + f[t_1, t_2, t_3]\, v_1 - f[t_1, t_2, t_3]\, t_1\, v_1 + f[t_1, t_2, t_3]\, t_1\, v_2 + t_2\, u_1\, f^{(0,1,0)}[t_1, t_2, t_3] - t_1\, t_2\, u_1\, f^{(0,1,0)}[t_1, t_2, t_3] - t_1\, u_1\, f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2\, u_1\, f^{(1,0,0)}[t_1, t_2, t_3],\ f[t_1, t_2, t_3]\, t_1\, u_1 + f[t_1, t_2, t_3]\, v_1 - f[t_1, t_2, t_3]\, t_1\, v_1 + f[t_1, t_2, t_3]\, t_1\, v_3 + t_3\, u_1\, f^{(0,0,1)}[t_1, t_2, t_3] - t_1\, t_3\, u_1\, f^{(0,0,1)}[t_1, t_2, t_3] - t_1\, u_1\, f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2\, u_1\, f^{(1,0,0)}[t_1, t_2, t_3],\ u_1,\ u_1 - t_1\, u_1 + t_1\, u_2,\ u_1 - t_1\, u_1 + t_1\, u_3\}$$

```
OCr = GPchecks // GP_{1,3} // GP_{1,2}; OCl - OCr
```

$$\{0, 0, 0, 0, 0, 0\}$$

Question. Does GP factor through G? How?

The Turbo-Gassner Representation

```
TG_{i_,j_}[ℱ_] := Expand[ℱ /. {
      f_ . v_{k_} :> Plus[f v_k /. v_j → (1 - t_i) v_i + t_i v_j,
        (1 - t_i^-1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f)
          (u_k /. u_j → (1 - t_i) u_i + t_i u_j) u_i w_j,
        Kδ_{k,i} f (u_j - u_i) u_i w_j],
      u_j → (1 - t_i) u_i + t_i u_j,
      w_i → w_i + (1 - t_i^-1) w_j,  w_j → t_i^-1 w_j}];
```

```
TGchecks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3};

Short[R3l = TGchecks // TG1,2 // TG1,3 // TG2,3, 10]
```

$$\Big\{f[t_1, t_2, t_3]\, v_1 - f[t_1, t_2, t_3]\, u_1^2\, w_2 + f[t_1, t_2, t_3]\, u_1\, u_2\, w_2 -$$
$$f[t_1, t_2, t_3]\, u_1^2\, w_3 + f[t_1, t_2, t_3]\, u_1\, u_3\, w_3 - \frac{t_3\, u_1^2\, w_3\, f^{(0,0,1)}[t_1, t_2, t_3]}{t_2} +$$
$$\frac{t_3\, u_1^2\, w_3\, f^{(0,0,1)}[t_1, t_2, t_3]}{t_1\, t_2} - t_3\, u_1\, u_2\, w_3\, f^{(0,0,1)}[t_1, t_2, t_3] +$$
$$\ll 9 \gg + \frac{t_2\, u_1^2\, w_3\, f^{(0,1,0)}[t_1, t_2, t_3]}{t_1} - u_1\, u_2\, w_3\, f^{(0,1,0)}[t_1, t_2, t_3] +$$
$$t_2\, u_1\, u_2\, w_3\, f^{(0,1,0)}[t_1, t_2, t_3] - u_1^2\, w_2\, f^{(1,0,0)}[t_1, t_2, t_3] + t_1\, u_1^2\, w_2\, f^{(1,0,0)}[t_1, t_2, t_3] -$$
$$u_1^2\, w_3\, f^{(1,0,0)}[t_1, t_2, t_3] + t_1\, u_1^2\, w_3\, f^{(1,0,0)}[t_1, t_2, t_3], \ll 7 \gg, \frac{w_3}{t_1\, t_2}\Big\}$$

```
R3r = TGchecks // TG2,3 // TG1,3 // TG1,2; R3l - R3r
```

$$\{0, 0, 0, 0, 0, 0, 0, 0\}$$

```
Short[OCl = TGchecks // TG1,2 // TG1,3]
```

$$\Big\{f[t_1, t_2, t_3]\, v_1 - \ll 1 \gg + \ll 15 \gg + t_1\, u_1^2\, w_3\, f^{(1,0,0)}[t_1, t_2, t_3], \ll 7 \gg, \frac{\ll 1 \gg}{\ll 1 \gg}\Big\}$$

```
OCr = TGchecks // TG1,3 // TG1,2; OCl - OCr
```

$$\{0, -f[t_1, t_2, t_3]\, u_1\, u_2\, w_3 + f[t_1, t_2, t_3]\, t_1\, u_1\, u_2\, w_3 + f[t_1, t_2, t_3]\, u_1\, u_3\, w_3 -$$
$$f[t_1, t_2, t_3]\, t_1\, u_1\, u_3\, w_3, -f[t_1, t_2, t_3]\, u_1\, u_2\, w_2 + f[t_1, t_2, t_3]\, t_1\, u_1\, u_2\, w_2 +$$
$$f[t_1, t_2, t_3]\, u_1\, u_3\, w_2 - f[t_1, t_2, t_3]\, t_1\, u_1\, u_3\, w_2, 0, 0, 0, 0, 0, 0\}$$

The Turbo-Burau Representation

```
η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
TBi_,j_[ξ_] :=
  Expand[ξ /. {
      f_. vk_ :> Plus[f vk /. vj → (1 - t - η[i]) vi + (t + η[i]) vj,
        (t - 1) (Coefficient[f, η[i]] - Coefficient[f, η[j]])
          (uk /. uj → (1 - t) ui + t uj) ui wj,
        Kδk,i (f /. _η → 0) (uj - ui) ui wj],
      uj → (1 - t) ui + t uj,
      wi → wi + (1 - t^-1) wj, wj → t^-1 wj}];
ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];
checks = {ff v1, ff v2, ff v3, u1^2 w1, u1^2 w2, u1, u2, u3, w1, w2, w3};
```

```
Short[R3l = checks // TB1,2 // TB1,3 // TB2,3, 10]
```

$$\left\{f_0\,v_1 - f_0\,u_1^2\,w_2 - f_1\,u_1^2\,w_2 + t\,f_1\,u_1^2\,w_2 + f_2\,u_1^2\,w_2 - t\,f_2\,u_1^2\,w_2 + f_0\,u_1\,u_2\,w_2 - f_0\,u_1^2\,w_3 - f_1\,u_1^2\,w_3 + t\,f_1\,u_1^2\,w_3 + 2\,f_2\,u_1^2\,w_3 - \frac{f_2\,u_1^2\,w_3}{t} - t\,f_2\,u_1^2\,w_3 - f_3\,u_1^2\,w_3 + \frac{f_3\,u_1^2\,w_3}{t} - f_2\,u_1\,u_2\,w_3 + t\,f_2\,u_1\,u_2\,w_3 + f_3\,u_1\,u_2\,w_3 - t\,f_3\,u_1\,u_2\,w_3 + f_0\,u_1\,u_3\,w_3 + f_1\,v_1\,\eta[1] + f_2\,v_1\,\eta[2] + f_3\,v_1\,\eta[3],\ \text{<<9>>},\ \frac{w_3}{t^2}\right\}$$

```
R3r = checks // TB2,3 // TB1,3 // TB1,2; R3l - R3r
```

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

```
Short[OCl = checks // TB1,2 // TB1,3]
```

$$\left\{\text{<<20>>} + f_2\,v_1\,\eta[2] + f_3\,v_1\,\eta[3],\ \text{<<54>>} + t\,f_3\,v_2\,\eta[3],\ \text{<<7>>},\ \frac{\text{<<1>>}}{t},\ \frac{w_3}{t}\right\}$$

```
OCr = checks // TB1,3 // TB1,2; OCl - OCr
```

$$\{0, -f_0\,u_1\,u_2\,w_3 + t\,f_0\,u_1\,u_2\,w_3 + f_0\,u_1\,u_3\,w_3 - t\,f_0\,u_1\,u_3\,w_3,$$
$$-f_0\,u_1\,u_2\,w_2 + t\,f_0\,u_1\,u_2\,w_2 + f_0\,u_1\,u_3\,w_2 - t\,f_0\,u_1\,u_3\,w_2, 0, 0, 0, 0, 0, 0, 0, 0\}$$