Pensieve header: Solving the equations for ρ_1 with imaginary Lagrangian (fail).

Preliminaries

(Alt) In[ ]:=

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Oaxaca-2210"];
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/la22/ap to compute rotation numbers.
```

The Old Program

(Alt) In[ ]:=

```
R1[s_, i_, j_] := s (g_{ji} (g_{j⁺,j} + g_{j,j⁺} - g_{ij}) - g_{ii} (g_{j,j⁺} - 1) - 1 / 2);
ρ[K_] := ρ[K] = Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
    {Cs, φ} = Rot[K]; n = Length[Cs];
    A = IdentityMatrix[2 n + 1];
    Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ((-T^s  T^s - 1)
                                                               (  0     -1  )))];
    Δ = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
    G = Inverse[A];
    ρ1 = Sum_{k=1}^{n} R1 @@ Cs[[k]] - Sum_{k=1}^{2 n} φ[[k]] (g_{kk} - 1 / 2);
    Factor@{Δ, Δ^2 ρ1 /. α_⁺ :> α + 1 /. g_{α_,β_} :> G[[α, β]]}];
```

The g-Rules

(Alt) In[ ]:=

```
δ_{i_,j_} := If[i === j, 1, 0];
gRules_{s_,i_,j_} := {g_{iβ_} :> δ_{iβ} + T^s g_{i⁺,β} + (1 - T^s) g_{j⁺,β}, g_{jβ_} :> δ_{jβ} + g_{j⁺,β},
  g_{α_,i} :> T^-s (g_{α,i⁺} - δ_{α,i⁺}), g_{α_,j} :> g_{α,j⁺} - (1 - T^s) g_{αi} - δ_{α,j⁺}}
(α_⁺)⁺ := α^"++"; (* this is for cosmetic reasons only *)
```

CF

(Alt) In[ ]:=

```
CF[ℰ_] := Module[{vs = Union[{ϵ}, Cases[ℰ, (g | p | x)__, ∞]]}, Total[
   CoefficientRules[Expand@ℰ, vs] /. (ps_ → c_) :> Factor[c] (Times @@ vs^ps)
  ]]
```

g2px and px2g

(Alt) In[ ]:=

```
g2px[ℰ_] := Module[{λ}, Expand[ℰ /. g_{α_,β_} :> λ p_α x_β] /. λ^k_. :> 1/k!]
```

(Alt) In[ ]:=

```
{p*, x*, π*, ξ*} = {π, ξ, p, x};  (u_{i_})* := (u*)_i;
```

(Alt) In[ ]:=

```
Zip_{}[ℰ_] := ℰ;
Zip_{ξ_,ξs___}[ℰ_] := (Collect[ℰ // Zip_{ξs}, ξ] /. f_. ξ^d_. :> (D[f, {ξ*, d}])) /. ξ* → 0
```

(Alt) In[ ]:=

```
px2g[ℰ_] := Module[{ps, xs, Q},
  ps = Union[Cases[ℰ, p_α_, ∞]];
  xs = Union[Cases[ℰ, x_α_, ∞]];
  Q = Sum[p0* x0* g_{p0[[2]],x0[[2]]}, {p0, ps}, {x0, xs}];
  Zip_{ps∪xs}[ℰ ℮^Q] // Expand
 ]
```

Generic Perturbations

(Alt) In[ ]:=

```
Module[{i, j, k},
 AllMonomials[{}, 0] = {1};
 AllMonomials[{}, d_Integer] /; d > 0 := {};
 AllMonomials[{v_, vs___}, d_Integer] :=
  Join @@ Table[v^(d-k) AllMonomials[{vs}, k], {k, 0, d}];
 AllMonomials[vs_List, {d_}] := Join @@ Table[AllMonomials[vs, k], {k, 0, d}];
 Basis[js_List, m_] := Flatten@Outer[Times,
    AllMonomials[Table[p_j, {j, js}], m], AllMonomials[Table[x_j, {j, js}], m]];
 Basis[js_List, {m_}] := Flatten@Table[Basis[js, k], {k, 0, m}];
 GenericCombination[bas_, c_] := bas.Table[c_j, {j, Length@bas}];
 GenericCombination[bas_, c_{k_}] := bas.Table[c_{k,j}, {j, Length@bas}];
]
```

(Alt) In[•]:=

```
Module[{k, x1, x2, p1, p2},
  r_d_[1, i_, j_] :=
   Expand@Together@Sum[ϵ^k GenericCombination[Basis[{i, j}, {k + 1}], ca_k], {k, d}];
  r_d_[-1, i_, j_] :=
   Expand@Together@Sum[ϵ^k GenericCombination[Basis[{i, j}, {k + 1}], cb_k], {k, d}];
  γ_d_[0, j_] := 0;
  γ_d_[1, j_] :=
   Expand@Together@Sum[ϵ^k GenericCombination[Basis[{j}, {k + 1}], cc_k], {k, d}];
  γ_d_[-1, j_] :=
   Expand@Together@Sum[ϵ^k GenericCombination[Basis[{j}, {k + 1}], cd_k], {k, d}];
  {x1*, x2*, p1*, p2*} = {p1, p2, x1, x2};
  r_d_[s_, φi_, φj_, i_, j_] := Normal[Log[O[ϵ]^(d+1) + Zip_{x1,x2}[Exp[O[ϵ]^(d+1) +
          (γ_d[φi, i] /. x_i → x_i + x1) +
          (γ_d[φj, j] /. x_j → x_j + x2) + (r_d[s, i, j] /. {p_i → p_i - p1, p_j → p_j - p2})
        ]]]]
 ];
```

(Alt) In[•]:=

```
r_1[1, j, k] // CF
```

(Alt) Out[•]=

$$\in ca_{1,1} + \in p_j x_j ca_{1,2} + \in p_j x_k ca_{1,3} + \in p_k x_j ca_{1,4} + \in p_k x_k ca_{1,5} +$$
$$\in p_j^2 x_j^2 ca_{1,6} + \in p_j^2 x_j x_k ca_{1,7} + \in p_j^2 x_k^2 ca_{1,8} + \in p_j p_k x_j^2 ca_{1,9} + \in p_j p_k x_j x_k ca_{1,10} +$$
$$\in p_j p_k x_k^2 ca_{1,11} + \in p_k^2 x_j^2 ca_{1,12} + \in p_k^2 x_j x_k ca_{1,13} + \in p_k^2 x_k^2 ca_{1,14}$$

(Alt) In[•]:=

```
r_2[-1, j, k] // CF
```

(Alt) Out[•]=

$$\in cb_{1,1} + \in p_j x_j cb_{1,2} + \in p_j x_k cb_{1,3} + \in p_k x_j cb_{1,4} + \in p_k x_k cb_{1,5} + \in p_j^2 x_j^2 cb_{1,6} + \in p_j^2 x_j x_k cb_{1,7} +$$
$$\in p_j^2 x_k^2 cb_{1,8} + \in p_j p_k x_j^2 cb_{1,9} + \in p_j p_k x_j x_k cb_{1,10} + \in p_j p_k x_k^2 cb_{1,11} + \in p_k^2 x_j^2 cb_{1,12} + \in p_k^2 x_j x_k cb_{1,13} +$$
$$\in p_k^2 x_k^2 cb_{1,14} + \in^2 cb_{2,1} + \in^2 p_j x_j cb_{2,2} + \in^2 p_j x_k cb_{2,3} + \in^2 p_k x_j cb_{2,4} + \in^2 p_k x_k cb_{2,5} +$$
$$\in^2 p_j^2 x_j^2 cb_{2,6} + \in^2 p_j^2 x_j x_k cb_{2,7} + \in^2 p_j^2 x_k^2 cb_{2,8} + \in^2 p_j p_k x_j^2 cb_{2,9} + \in^2 p_j p_k x_j x_k cb_{2,10} +$$
$$\in^2 p_j p_k x_k^2 cb_{2,11} + \in^2 p_k^2 x_j^2 cb_{2,12} + \in^2 p_k^2 x_j x_k cb_{2,13} + \in^2 p_k^2 x_k^2 cb_{2,14} + \in^2 p_j^3 x_j^3 cb_{2,15} +$$
$$\in^2 p_j^3 x_j^2 x_k cb_{2,16} + \in^2 p_j^3 x_j x_k^2 cb_{2,17} + \in^2 p_j^3 x_k^3 cb_{2,18} + \in^2 p_j^2 p_k x_j^3 cb_{2,19} + \in^2 p_j^2 p_k x_j^2 x_k cb_{2,20} +$$
$$\in^2 p_j^2 p_k x_j x_k^2 cb_{2,21} + \in^2 p_j^2 p_k x_k^3 cb_{2,22} + \in^2 p_j p_k^2 x_j^3 cb_{2,23} + \in^2 p_j p_k^2 x_j^2 x_k cb_{2,24} + \in^2 p_j p_k^2 x_j x_k^2 cb_{2,25} +$$
$$\in^2 p_j p_k^2 x_k^3 cb_{2,26} + \in^2 p_k^3 x_j^3 cb_{2,27} + \in^2 p_k^3 x_j^2 x_k cb_{2,28} + \in^2 p_k^3 x_j x_k^2 cb_{2,29} + \in^2 p_k^3 x_k^3 cb_{2,30}$$

(Alt) In[•]:=

```
γ_2[1, i] // CF
```

(Alt) Out[•]=

$$\in cc_{1,1} + \in p_i^2 x_i^2 cc_{1,2} + \in^2 cc_{2,1} + \in^2 p_i^2 x_i^2 cc_{2,2}$$

(Alt) In[•]:=
```
r2[-1, φj, φk, j, k] // CF
```
(Alt) Out[•]=
$$\text{Log}\left[e^{\gamma_2[\varphi j,j]+\gamma_2[\varphi k,k]}\right] + \epsilon\, cb_{1,1} + \epsilon\, p_j^2 x_j^2 cb_{1,2} + \epsilon\, p_j^2 x_j x_k cb_{1,3} + \epsilon\, p_j^2 x_k^2 cb_{1,4} + \epsilon\, p_j p_k x_j^2 cb_{1,5} + \epsilon\, p_j p_k x_j x_k cb_{1,6} + \epsilon\, p_j p_k x_k^2 cb_{1,7} + \epsilon\, p_k^2 x_j^2 cb_{1,8} + \epsilon\, p_k^2 x_j x_k cb_{1,9} + \epsilon\, p_k^2 x_k^2 cb_{1,10} + \epsilon^2 cb_{2,1} + \epsilon^2 p_j^2 x_j^2 cb_{2,2} + \epsilon^2 p_j^2 x_j x_k cb_{2,3} + \epsilon^2 p_j^2 x_k^2 cb_{2,4} + \epsilon^2 p_j p_k x_j^2 cb_{2,5} + \epsilon^2 p_j p_k x_j x_k cb_{2,6} + \epsilon^2 p_j p_k x_k^2 cb_{2,7} + \epsilon^2 p_k^2 x_j^2 cb_{2,8} + \epsilon^2 p_k^2 x_j x_k cb_{2,9} + \epsilon^2 p_k^2 x_k^2 cb_{2,10}$$

Non-Universally Solving at d=1

(Alt) In[•]:=
```
d = 1;
vars =
 Cases[Variables[r_d[1, i1, j1] + r_d[-1, i2, j2] + γ_d[1, k1] + γ_d[-1, k2]], (ca | cb | cc | cd)__]
```
(Alt) Out[•]=
$$\{ca_{1,1}, ca_{1,2}, ca_{1,3}, ca_{1,4}, ca_{1,5}, ca_{1,6}, ca_{1,7}, ca_{1,8}, ca_{1,9}, ca_{1,10}, ca_{1,11}, ca_{1,12}, ca_{1,13}, ca_{1,14}, cb_{1,1}, cb_{1,2}, cb_{1,3}, cb_{1,4}, cb_{1,5}, cb_{1,6}, cb_{1,7}, cb_{1,8}, cb_{1,9}, cb_{1,10}, cb_{1,11}, cb_{1,12}, cb_{1,13}, cb_{1,14}, cc_{1,1}, cc_{1,2}, cc_{1,3}, cd_{1,1}, cd_{1,2}, cd_{1,3}\}$$

$C\,\overline{C}$

(Alt) In[•]:=
```
lhs = Module[{x1, p1},
   {x1*, p1*} = {p1, x1};
   Normal[Log[
     O[ϵ]^(d+1) + Zip_{x1}[Exp[O[ϵ]^(d+1) + (γ_d[1, i] /. x_i → x_i + x1) + (γ_d[-1, i] /. p_i → p_i - p1)]]]]
  ]
rhs = 0
```
(Alt) Out[•]=
$$\epsilon\left(cc_{1,1} + p_i x_i cc_{1,2} + p_i^2 x_i^2 cc_{1,3} + cd_{1,1} + p_i x_i cd_{1,2} + p_i^2 x_i^2 cd_{1,3}\right)$$

(Alt) Out[•]=
$$0$$

(Alt) In[•]:=
```
covars = DeleteCases[Variables[lhs - rhs], T | (ca | cb | cc | cd)__]
```
(Alt) Out[•]=
$$\{\epsilon, p_i, x_i\}$$

(Alt) In[•]:=
```
eqnsCCbar = (# == 0) & /@ Union[Last /@ CoefficientRules[Expand[lhs - rhs], covars]]
```
(Alt) Out[•]=
$$\{cc_{1,1} + cd_{1,1} == 0,\ cc_{1,2} + cd_{1,2} == 0,\ cc_{1,3} + cd_{1,3} == 0\}$$

(Alt) In[•]:=
```
vars = Cases[
  Variables[r_d[1, i1, j1] + r_d[-1, i2, j2] + γ_d[1, k1] + γ_d[-1, k2]], (ca | cb | cc | cd)__]
{sol} = Solve[eqnsCCbar, vars]
sol /. Rule → Set;
γ_d[1, k]
γ_d[-1, k]
```
(Alt) Out[•]=
$$\{ca_{1,1}, ca_{1,2}, ca_{1,3}, ca_{1,4}, ca_{1,5}, ca_{1,6}, ca_{1,7}, ca_{1,8}, ca_{1,9}, ca_{1,10}, ca_{1,11}, ca_{1,12}, ca_{1,13}, ca_{1,14}, cb_{1,1}, cb_{1,2}, cb_{1,3}, cb_{1,4}, cb_{1,5}, cb_{1,6}, cb_{1,7}, cb_{1,8}, cb_{1,9}, cb_{1,10}, cb_{1,11}, cb_{1,12}, cb_{1,13}, cb_{1,14}, cc_{1,1}, cc_{1,2}, cc_{1,3}, cd_{1,1}, cd_{1,2}, cd_{1,3}\}$$

Solve: Equations may not give solutions for all "solve" variables.

(Alt) Out[•]=
$$\{\{cd_{1,1} \to -cc_{1,1}, cd_{1,2} \to -cc_{1,2}, cd_{1,3} \to -cc_{1,3}\}\}$$

(Alt) Out[•]=
$$\in cc_{1,1} + \in p_k x_k cc_{1,2} + \in p_k^2 x_k^2 cc_{1,3}$$

(Alt) Out[•]=
$$-\in cc_{1,1} - \in p_k x_k cc_{1,2} - \in p_k^2 x_k^2 cc_{1,3}$$

R3

(Alt) In[•]:=
```
Short[lhs = CF[Module[{es = {i, j, k, i⁺, j⁺, k⁺}},
    Times[
      Normal@Series[Exp[r_d[1, j, k] + r_d[1, i, k⁺] + r_d[1, i⁺, j⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
    ] // Zip[(p_#&/@es)∪(x_#&/@es)] // Expand
  ] //. gRules_{1,j,k} ∪ gRules_{1,i,k⁺} ∪ gRules_{1,i⁺,j⁺}]]
```
(Alt) Out[•]//Short=
$$1 + \text{<<64>>} + 4 \in \left(ca_{1,8} - 2\,T\,ca_{1,8} + T^2\,ca_{\text{<<1>>}} + ca_{\text{<<1>>}} - T\,ca_{1,11} + ca_{1,14}\right) g^2_{k^{++},k^{++}}$$

(Alt) In[•]:=
```
Short[rhs = CF[Module[{es = {i, j, k, i⁺, j⁺, k⁺}},
    Times[
      Normal@Series[Exp[r_d[1, i, j] + r_d[1, i⁺, k] + r_d[1, j⁺, k⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
    ] // Zip[(p_#&/@es)∪(x_#&/@es)] // Expand
  ] //. gRules_{1,i,j} ∪ gRules_{1,i⁺,k} ∪ gRules_{1,j⁺,k⁺}]]
```
(Alt) Out[•]//Short=
$$1 + \text{<<61>>} + 4 \in \left(ca_{1,8} - 2\,T\,ca_{1,8} + T^2\,ca_{\text{<<1>>}} + ca_{\text{<<1>>}} - T\,ca_{1,11} + ca_{1,14}\right) g^2_{k^{++},k^{++}}$$

(Alt) In[•]:=
```
me = Exponent[lhs - rhs, T, Min]
```
(Alt) Out[•]=
$$-4$$

(Alt) In[•]:=
```
covars = DeleteCases[Variables[lhs - rhs], T | (ca | cb | cc | cd)__]
```

(Alt) Out[•]=
$$\{\in, g_{i^{++},i^{++}}, g_{i^{++},j^{++}}, g_{i^{++},k^{++}}, g_{j^{++},i^{++}}, g_{j^{++},j^{++}}, g_{j^{++},k^{++}}, g_{k^{++},i^{++}}, g_{k^{++},j^{++}}, g_{k^{++},k^{++}}\}$$

(Alt) In[•]:=
```
Short[
 eqnsR3 = (# == 0) & /@ Union[Last /@ CoefficientRules[Expand[T^-me (lhs - rhs)], covars]]]
```

(Alt) Out[•]//Short=
$$\left\{-T^3\, ca_{1,3} + T^4\, ca_{1,3} == 0,\ \ll 31 \gg,\ -4\, T^3\, ca_{1,8} + \ll 10 \gg + 4\, T^4\, ca_{1,14} == 0\right\}$$

R2b

(Alt) In[ ]:=

```
lhs = CF[Module[{es = {i, j, i⁺, j⁺}},
    Times[
      Normal@Series[Exp[r_d[1, i, j] + r_d[-1, i⁺, j⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
     ] // Zip_{(p_#&/@es)∪(x_#&/@es)} // Expand
  ] //. gRules_{1,i,j} ∪ gRules_{-1,i⁺,j⁺}]
```

(Alt) Out[ ]=

$$1+\epsilon\left(ca_{1,1}+cb_{1,1}\right)+\frac{\epsilon\left(T\,ca_{1,2}+T\,cb_{1,2}+cb_{1,3}-T\,cb_{1,3}\right)g_{i^{++},i^{++}}}{T}+$$
$$\frac{2\epsilon\left(T^2\,ca_{1,6}+T^2\,cb_{1,6}+T\,cb_{1,7}-T^2\,cb_{1,7}+cb_{1,8}-2\,T\,cb_{1,8}+T^2\,cb_{1,8}\right)g^2_{i^{++},i^{++}}}{T^2}+$$
$$\frac{\epsilon\left(T\,ca_{1,3}+cb_{1,3}\right)g_{i^{++},j^{++}}}{T}+\frac{2\epsilon\left(T^2\,ca_{1,7}+T\,cb_{1,7}+2\,cb_{1,8}-2\,T\,cb_{1,8}\right)g_{i^{++},i^{++}}\,g_{i^{++},j^{++}}}{T^2}+$$
$$\frac{2\epsilon\left(T^2\,ca_{1,8}+cb_{1,8}\right)g^2_{i^{++},j^{++}}}{T^2}+$$
$$\frac{\epsilon\left(T\,ca_{1,4}-T\,cb_{1,2}+T^2\,cb_{1,2}-cb_{1,3}+2\,T\,cb_{1,3}-T^2\,cb_{1,3}+T^2\,cb_{1,4}+T\,cb_{1,5}-T^2\,cb_{1,5}\right)g_{j^{++},i^{++}}}{T}+\frac{1}{T^2}$$
$$2\epsilon\left(T^2\,ca_{1,9}-2\,T^2\,cb_{1,6}+2\,T^3\,cb_{1,6}-2\,T\,cb_{1,7}+4\,T^2\,cb_{1,7}-2\,T^3\,cb_{1,7}-2\,cb_{1,8}+6\,T\,cb_{1,8}-6\,T^2\,cb_{1,8}+\right.$$
$$\left.2\,T^3\,cb_{1,8}+T^3\,cb_{1,9}+T^2\,cb_{1,10}-T^3\,cb_{1,10}+T\,cb_{1,11}-2\,T^2\,cb_{1,11}+T^3\,cb_{1,11}\right)g_{i^{++},i^{++}}\,g_{j^{++},i^{++}}+\frac{1}{T^2}$$
$$\epsilon\left(T^2\,ca_{1,10}-2\,T\,cb_{1,7}+2\,T^2\,cb_{1,7}-4\,cb_{1,8}+8\,T\,cb_{1,8}-4\,T^2\,cb_{1,8}+T^2\,cb_{1,10}+2\,T\,cb_{1,11}-2\,T^2\,cb_{1,11}\right)$$
$$g_{i^{++},j^{++}}\,g_{j^{++},i^{++}}+\frac{1}{T^2}\,2\epsilon\left(T^2\,ca_{1,12}+T^2\,cb_{1,6}-2\,T^3\,cb_{1,6}+T^4\,cb_{1,6}+T\,cb_{1,7}-3\,T^2\,cb_{1,7}+\right.$$
$$3\,T^3\,cb_{1,7}-T^4\,cb_{1,7}+cb_{1,8}-4\,T\,cb_{1,8}+6\,T^2\,cb_{1,8}-4\,T^3\,cb_{1,8}+T^4\,cb_{1,8}-T^3\,cb_{1,9}+$$
$$T^4\,cb_{1,9}-T^2\,cb_{1,10}+2\,T^3\,cb_{1,10}-T^4\,cb_{1,10}-T\,cb_{1,11}+3\,T^2\,cb_{1,11}-3\,T^3\,cb_{1,11}+$$
$$\left.T^4\,cb_{1,11}+T^4\,cb_{1,12}+T^3\,cb_{1,13}-T^4\,cb_{1,13}+T^2\,cb_{1,14}-2\,T^3\,cb_{1,14}+T^4\,cb_{1,14}\right)g^2_{j^{++},i^{++}}+$$
$$\frac{\epsilon\left(T\,ca_{1,5}-cb_{1,3}+T\,cb_{1,3}+T\,cb_{1,5}\right)g_{j^{++},j^{++}}}{T}+\frac{1}{T^2}$$
$$\epsilon\left(T^2\,ca_{1,10}-2\,T\,cb_{1,7}+2\,T^2\,cb_{1,7}-4\,cb_{1,8}+8\,T\,cb_{1,8}-4\,T^2\,cb_{1,8}+T^2\,cb_{1,10}+2\,T\,cb_{1,11}-2\,T^2\,cb_{1,11}\right)$$
$$g_{i^{++},i^{++}}\,g_{j^{++},j^{++}}+\frac{2\epsilon\left(T^2\,ca_{1,11}-2\,cb_{1,8}+2\,T\,cb_{1,8}+T\,cb_{1,11}\right)g_{i^{++},j^{++}}\,g_{j^{++},j^{++}}}{T^2}+$$
$$\frac{1}{T^2}\,2\epsilon\left(T^2\,ca_{1,13}+T\,cb_{1,7}-2\,T^2\,cb_{1,7}+T^3\,cb_{1,7}+2\,cb_{1,8}-6\,T\,cb_{1,8}+6\,T^2\,cb_{1,8}-2\,T^3\,cb_{1,8}-T^2\,cb_{1,10}+\right.$$
$$\left.T^3\,cb_{1,10}-2\,T\,cb_{1,11}+4\,T^2\,cb_{1,11}-2\,T^3\,cb_{1,11}+T^3\,cb_{1,13}+2\,T^2\,cb_{1,14}-2\,T^3\,cb_{1,14}\right)g_{j^{++},i^{++}}\,g_{j^{++},j^{++}}+$$
$$\frac{2\epsilon\left(T^2\,ca_{1,14}+cb_{1,8}-2\,T\,cb_{1,8}+T^2\,cb_{1,8}-T\,cb_{1,11}+T^2\,cb_{1,11}+T^2\,cb_{1,14}\right)g^2_{j^{++},j^{++}}}{T^2}$$

(Alt) In[•]:=
```
rhs = 1
```
(Alt) Out[•]=
```
1
```

(Alt) In[•]:=
```
me = Exponent[lhs - rhs, T, Min]
```
(Alt) Out[•]=
```
-2
```

(Alt) In[•]:=
```
covars = DeleteCases[Variables[lhs - rhs], T | (ca | cb | cc | cd)__]
```
(Alt) Out[•]=
$$\{\in, g_{i^{++},i^{++}}, g_{i^{++},j^{++}}, g_{j^{++},i^{++}}, g_{j^{++},j^{++}}\}$$

(Alt) In[•]:=
```
eqnsR2b =
 (Factor[#] == 0) & /@ Union[Last /@ CoefficientRules[Expand[T^-me (lhs - rhs)], covars]]
```
(Alt) Out[•]=
$$\Big\{T^2 (ca_{1,1} + cb_{1,1}) == 0,\ T (T\, ca_{1,3} + cb_{1,3}) == 0,$$
$$T (T\, ca_{1,2} + T\, cb_{1,2} + cb_{1,3} - T\, cb_{1,3}) == 0,\ T (T\, ca_{1,5} - cb_{1,3} + T\, cb_{1,3} + T\, cb_{1,5}) == 0,$$
$$T \left(T\, ca_{1,4} - T\, cb_{1,2} + T^2\, cb_{1,2} - cb_{1,3} + 2\, T\, cb_{1,3} - T^2\, cb_{1,3} + T^2\, cb_{1,4} + T\, cb_{1,5} - T^2\, cb_{1,5}\right) == 0,$$
$$2 \left(T^2\, ca_{1,8} + cb_{1,8}\right) == 0,\ 2 \left(T^2\, ca_{1,7} + T\, cb_{1,7} + 2\, cb_{1,8} - 2\, T\, cb_{1,8}\right) == 0,$$
$$2 \left(T^2\, ca_{1,6} + T^2\, cb_{1,6} + T\, cb_{1,7} - T^2\, cb_{1,7} + cb_{1,8} - 2\, T\, cb_{1,8} + T^2\, cb_{1,8}\right) == 0,$$
$$2 \left(T^2\, ca_{1,11} - 2\, cb_{1,8} + 2\, T\, cb_{1,8} + T\, cb_{1,11}\right) == 0,$$
$$T^2\, ca_{1,10} - 2\, T\, cb_{1,7} + 2\, T^2\, cb_{1,7} - 4\, cb_{1,8} + 8\, T\, cb_{1,8} - 4\, T^2\, cb_{1,8} + T^2\, cb_{1,10} + 2\, T\, cb_{1,11} - 2\, T^2\, cb_{1,11} == 0,$$
$$2 \left(T^2\, ca_{1,9} - 2\, T^2\, cb_{1,6} + 2\, T^3\, cb_{1,6} - 2\, T\, cb_{1,7} + 4\, T^2\, cb_{1,7} - 2\, T^3\, cb_{1,7} - 2\, cb_{1,8} + 6\, T\, cb_{1,8} - 6\, T^2\, cb_{1,8} + 2\, T^3\, cb_{1,8} + T^3\, cb_{1,9} + T^2\, cb_{1,10} - T^3\, cb_{1,10} + T\, cb_{1,11} - 2\, T^2\, cb_{1,11} + T^3\, cb_{1,11}\right) == 0,$$
$$2 \left(T^2\, ca_{1,14} + cb_{1,8} - 2\, T\, cb_{1,8} + T^2\, cb_{1,8} - T\, cb_{1,11} + T^2\, cb_{1,11} + T^2\, cb_{1,14}\right) == 0,$$
$$2 \left(T^2\, ca_{1,13} + T\, cb_{1,7} - 2\, T^2\, cb_{1,7} + T^3\, cb_{1,7} + 2\, cb_{1,8} - 6\, T\, cb_{1,8} + 6\, T^2\, cb_{1,8} - 2\, T^3\, cb_{1,8} - T^2\, cb_{1,10} + T^3\, cb_{1,10} - 2\, T\, cb_{1,11} + 4\, T^2\, cb_{1,11} - 2\, T^3\, cb_{1,11} + T^3\, cb_{1,13} + 2\, T^2\, cb_{1,14} - 2\, T^3\, cb_{1,14}\right) == 0,$$
$$2 \left(T^2\, ca_{1,12} + T^2\, cb_{1,6} - 2\, T^3\, cb_{1,6} + T^4\, cb_{1,6} + T\, cb_{1,7} - 3\, T^2\, cb_{1,7} + 3\, T^3\, cb_{1,7} - T^4\, cb_{1,7} + cb_{1,8} - 4\, T\, cb_{1,8} + 6\, T^2\, cb_{1,8} - 4\, T^3\, cb_{1,8} + T^4\, cb_{1,8} - T^3\, cb_{1,9} + T^4\, cb_{1,9} - T^2\, cb_{1,10} + 2\, T^3\, cb_{1,10} - T^4\, cb_{1,10} - T\, cb_{1,11} + 3\, T^2\, cb_{1,11} - 3\, T^3\, cb_{1,11} + T^4\, cb_{1,11} + T^4\, cb_{1,12} + T^3\, cb_{1,13} - T^4\, cb_{1,13} + T^2\, cb_{1,14} - 2\, T^3\, cb_{1,14} + T^4\, cb_{1,14}\right) == 0\Big\}$$

R2c

(Alt) In[•]:=

```
lhs = CF[Module[{es = {i, j, i⁺, j⁺}},
    Times[
      Normal@Series[Exp[r_d[-1, i, j⁺] + r_d[1, i⁺, j] + γ_d[1, j⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
     ] // Zip_{(p_#&/@es)∪(x_#&/@es)} // Expand
   ] //. gRules_{-1,i,j⁺} ∪ gRules_{1,i⁺,j}]
```

(Alt) Out[•]=

$$1+\epsilon\left(ca_{1,1}-ca_{1,3}+T\,ca_{1,3}+2\,ca_{1,8}-4\,T\,ca_{1,8}+2\,T^2\,ca_{1,8}+cb_{1,1}+cc_{1,1}\right)+$$
$$\frac{\epsilon\left(T\,ca_{1,2}-2\,T\,ca_{1,7}+2\,T^2\,ca_{1,7}+T\,cb_{1,2}+cb_{1,3}-T\,cb_{1,3}\right)g_{i^{++},i^{++}}}{T}+$$
$$\frac{2\,\epsilon\left(T^2\,ca_{1,6}+T^2\,cb_{1,6}+T\,cb_{1,7}-T^2\,cb_{1,7}+cb_{1,8}-2\,T\,cb_{1,8}+T^2\,cb_{1,8}\right)g^2_{i^{++},i^{++}}}{T^2}+$$
$$\epsilon\left(T\,ca_{1,3}-4\,T\,ca_{1,8}+4\,T^2\,ca_{1,8}+cb_{1,3}\right)g_{i^{++},j^{++}}+$$
$$\frac{2\,\epsilon\left(T^2\,ca_{1,7}+T\,cb_{1,7}+2\,cb_{1,8}-2\,T\,cb_{1,8}\right)g_{i^{++},i^{++}}\,g_{i^{++},j^{++}}}{T}+$$
$$2\,\epsilon\left(T^2\,ca_{1,8}+cb_{1,8}\right)g^2_{i^{++},j^{++}}-\frac{1}{T}\epsilon\left(-ca_{1,2}+T\,ca_{1,2}-ca_{1,4}+2\,ca_{1,7}-4\,T\,ca_{1,7}+\right.$$
$$\left.2\,T^2\,ca_{1,7}+ca_{1,10}-T\,ca_{1,10}-T\,cb_{1,4}-cb_{1,5}+T\,cb_{1,5}-cc_{1,2}+T\,cc_{1,2}\right)g_{j^{++},i^{++}}-\frac{1}{T^2}$$
$$2\,\epsilon\left(-2\,T\,ca_{1,6}+2\,T^2\,ca_{1,6}-T\,ca_{1,9}-T^2\,cb_{1,9}-T\,cb_{1,10}+T^2\,cb_{1,10}-cb_{1,11}+2\,T\,cb_{1,11}-T^2\,cb_{1,11}\right)$$
$$g_{i^{++},i^{++}}\,g_{j^{++},i^{++}}-\frac{\epsilon\left(-2\,T\,ca_{1,7}+2\,T^2\,ca_{1,7}-T\,ca_{1,10}-T\,cb_{1,10}-2\,cb_{1,11}+2\,T\,cb_{1,11}\right)g_{i^{++},j^{++}}\,g_{j^{++},i^{++}}}{T}+$$
$$\frac{1}{T^2}\,2\,\epsilon\left(ca_{1,6}-2\,T\,ca_{1,6}+T^2\,ca_{1,6}+ca_{1,9}-T\,ca_{1,9}+ca_{1,12}+T^2\,cb_{1,12}+T\,cb_{1,13}-\right.$$
$$\left.T^2\,cb_{1,13}+cb_{1,14}-2\,T\,cb_{1,14}+T^2\,cb_{1,14}+cc_{1,3}-2\,T\,cc_{1,3}+T^2\,cc_{1,3}\right)g^2_{j^{++},i^{++}}+$$
$$\epsilon\left(ca_{1,3}-T\,ca_{1,3}+ca_{1,5}-4\,ca_{1,8}+8\,T\,ca_{1,8}-4\,T^2\,ca_{1,8}-2\,ca_{1,11}+2\,T\,ca_{1,11}+cb_{1,5}+cc_{1,2}\right)g_{j^{++},j^{++}}-$$
$$\frac{\epsilon\left(-2\,T\,ca_{1,7}+2\,T^2\,ca_{1,7}-T\,ca_{1,10}-T\,cb_{1,10}-2\,cb_{1,11}+2\,T\,cb_{1,11}\right)g_{i^{++},i^{++}}\,g_{j^{++},j^{++}}}{T}-$$
$$2\,\epsilon\left(-2\,T\,ca_{1,8}+2\,T^2\,ca_{1,8}-T\,ca_{1,11}-cb_{1,11}\right)g_{i^{++},j^{++}}\,g_{j^{++},j^{++}}+$$
$$\frac{1}{T}\,2\,\epsilon\left(ca_{1,7}-2\,T\,ca_{1,7}+T^2\,ca_{1,7}+ca_{1,10}-T\,ca_{1,10}+ca_{1,13}+\right.$$
$$\left.T\,cb_{1,13}+2\,cb_{1,14}-2\,T\,cb_{1,14}+2\,cc_{1,3}-2\,T\,cc_{1,3}\right)g_{j^{++},i^{++}}\,g_{j^{++},j^{++}}+$$
$$2\,\epsilon\left(ca_{1,8}-2\,T\,ca_{1,8}+T^2\,ca_{1,8}+ca_{1,11}-T\,ca_{1,11}+ca_{1,14}+cb_{1,14}+cc_{1,3}\right)g^2_{j^{++},j^{++}}$$

(Alt) In[•]:=
```
rhs = CF[Module[{es = {(j^+)^+}},
   Times[
     Normal@Series[Exp[γ_d[1, (j^+)^+]], {ϵ, 0, d}],
     Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
    ] // Zip_{(p_#&/@es)∪(x_#&/@es)} // Expand
  ]]
```
(Alt) Out[•]=
$$1 + \epsilon\, cc_{1,1} + \epsilon\, cc_{1,2}\, g_{j^{++},j^{++}} + 2\,\epsilon\, cc_{1,3}\, g^2_{j^{++},j^{++}}$$

(Alt) In[•]:=
```
me = Exponent[lhs - rhs, T, Min]
```
(Alt) Out[•]=
$$-2$$

(Alt) In[•]:=
```
covars = DeleteCases[Variables[lhs - rhs], T | (ca | cb | cc | cd)__]
```
(Alt) Out[•]=
$$\{\epsilon,\ g_{i^{++},i^{++}},\ g_{i^{++},j^{++}},\ g_{j^{++},i^{++}},\ g_{j^{++},j^{++}}\}$$

(Alt) In[•]:=
```
eqnsR2c =
  (Factor[#] == 0) & /@ Union[Last /@ CoefficientRules[Expand[T^-me (lhs - rhs)], covars]]
```
(Alt) Out[•]=
$$\Big\{T^2 \left(ca_{1,1} - ca_{1,3} + T\, ca_{1,3} + 2\, ca_{1,8} - 4\, T\, ca_{1,8} + 2\, T^2\, ca_{1,8} + cb_{1,1}\right) == 0,$$
$$T \left(T\, ca_{1,2} - 2\, T\, ca_{1,7} + 2\, T^2\, ca_{1,7} + T\, cb_{1,2} + cb_{1,3} - T\, cb_{1,3}\right) == 0,$$
$$T^2 \left(T\, ca_{1,3} - 4\, T\, ca_{1,8} + 4\, T^2\, ca_{1,8} + cb_{1,3}\right) == 0,$$
$$-T^2 \left(-ca_{1,3} + T\, ca_{1,3} - ca_{1,5} + 4\, ca_{1,8} - 8\, T\, ca_{1,8} + 4\, T^2\, ca_{1,8} + 2\, ca_{1,11} - 2\, T\, ca_{1,11} - cb_{1,5}\right) == 0,$$
$$2\, T \left(T^2\, ca_{1,7} + T\, cb_{1,7} + 2\, cb_{1,8} - 2\, T\, cb_{1,8}\right) == 0,\ 2\, T^2 \left(T^2\, ca_{1,8} + cb_{1,8}\right) == 0,$$
$$2 \left(T^2\, ca_{1,6} + T^2\, cb_{1,6} + T\, cb_{1,7} - T^2\, cb_{1,7} + cb_{1,8} - 2\, T\, cb_{1,8} + T^2\, cb_{1,8}\right) == 0,$$
$$-T \left(-2\, T\, ca_{1,7} + 2\, T^2\, ca_{1,7} - T\, ca_{1,10} - T\, cb_{1,10} - 2\, cb_{1,11} + 2\, T\, cb_{1,11}\right) == 0,$$
$$-2\, T^2 \left(-2\, T\, ca_{1,8} + 2\, T^2\, ca_{1,8} - T\, ca_{1,11} - cb_{1,11}\right) == 0,$$
$$-2 \left(-2\, T\, ca_{1,6} + 2\, T^2\, ca_{1,6} - T\, ca_{1,9} - T^2\, cb_{1,9} - T\, cb_{1,10} + T^2\, cb_{1,10} - cb_{1,11} + 2\, T\, cb_{1,11} - T^2\, cb_{1,11}\right) == 0,$$
$$2\, T^2 \left(ca_{1,8} - 2\, T\, ca_{1,8} + T^2\, ca_{1,8} + ca_{1,11} - T\, ca_{1,11} + ca_{1,14} + cb_{1,14}\right) == 0,$$
$$-T \left(-ca_{1,2} + T\, ca_{1,2} - ca_{1,4} + 2\, ca_{1,7} - 4\, T\, ca_{1,7} + 2\, T^2\, ca_{1,7} + ca_{1,10} - T\, ca_{1,10} - T\, cb_{1,4} - cb_{1,5} + T\, cb_{1,5} - cc_{1,2} + T\, cc_{1,2}\right) == 0,\ 2\, T \left(ca_{1,7} - 2\, T\, ca_{1,7} + T^2\, ca_{1,7} + ca_{1,10} - T\, ca_{1,10} + ca_{1,13} + T\, cb_{1,13} + 2\, cb_{1,14} - 2\, T\, cb_{1,14} + 2\, cc_{1,3} - 2\, T\, cc_{1,3}\right) == 0,$$
$$2 \left(ca_{1,6} - 2\, T\, ca_{1,6} + T^2\, ca_{1,6} + ca_{1,9} - T\, ca_{1,9} + ca_{1,12} + T^2\, cb_{1,12} + T\, cb_{1,13} - T^2\, cb_{1,13} + cb_{1,14} - 2\, T\, cb_{1,14} + T^2\, cb_{1,14} + cc_{1,3} - 2\, T\, cc_{1,3} + T^2\, cc_{1,3}\right) == 0\Big\}$$

R1l

(Alt) In[•]:=
```
lhs = CF[Module[{es = {i, i⁺}},
    Times[
      Normal@Series[Exp[r_d[1, i⁺, i] + γ_d[1, i⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
    ] // Zip_{(p_#&/@es)∪(x_#&/@es)} // Expand
  ] //. {g_{i⁺,β_} :> T^-1 δ_{i⁺,β} + g_{i⁺⁺,β}, g_{i,β_} :> δ_{i,β} + g_{i⁺,β}}]
```

(Alt) Out[•]=
$$1+\frac{1}{T^2}\in\left(T^2\,ca_{1,1}+T\,ca_{1,2}+T\,ca_{1,4}+T^2\,ca_{1,5}+2\,ca_{1,6}+2\,ca_{1,9}+T\,ca_{1,10}+2\,ca_{1,12}+2\,T\,ca_{1,13}+2\,T^2\,ca_{1,14}+T^2\,cc_{1,1}+T\,cc_{1,2}+2\,cc_{1,3}\right)+\frac{\in\left(T\,ca_{1,3}+T\,ca_{1,5}+2\,ca_{1,7}+2\,ca_{1,10}+2\,T\,ca_{1,11}+2\,ca_{1,13}+4\,T\,ca_{1,14}\right)g_{i^{++},i}}{T}+2\in\left(ca_{1,8}+ca_{1,11}+ca_{1,14}\right)g_{i^{++},i}^2+\frac{\in\left(T\,ca_{1,2}+T\,ca_{1,4}+4\,ca_{1,6}+4\,ca_{1,9}+T\,ca_{1,10}+4\,ca_{1,12}+2\,T\,ca_{1,13}+T\,cc_{1,2}+4\,cc_{1,3}\right)g_{i^{++},i^+}}{T}+2\in\left(ca_{1,7}+ca_{1,10}+ca_{1,13}\right)g_{i^{++},i}\,g_{i^{++},i^+}+2\in\left(ca_{1,6}+ca_{1,9}+ca_{1,12}+cc_{1,3}\right)g_{i^{++},i^+}^2$$

(Alt) In[•]:=
```
rhs = 1
```

(Alt) Out[•]=
$$1$$

(Alt) In[•]:=
```
me = Exponent[lhs - rhs, T, Min]
```

(Alt) Out[•]=
$$-2$$

(Alt) In[•]:=
```
covars = DeleteCases[Variables[lhs - rhs], T | (ca | cb | cc | cd)__]
```

(Alt) Out[•]=
$$\{\in,\ g_{i^{++},i},\ g_{i^{++},i^+}\}$$

(Alt) In[•]:=
```
eqnsR1l =
  (Factor[#] == 0) & /@ Union[Last /@ CoefficientRules[Expand[T^-me (lhs - rhs)], covars]]
```

(Alt) Out[•]=
$$\left\{2\,T^2\left(ca_{1,7}+ca_{1,10}+ca_{1,13}\right)==0,\ 2\,T^2\left(ca_{1,8}+ca_{1,11}+ca_{1,14}\right)==0,\ T\left(T\,ca_{1,3}+T\,ca_{1,5}+2\,ca_{1,7}+2\,ca_{1,10}+2\,T\,ca_{1,11}+2\,ca_{1,13}+4\,T\,ca_{1,14}\right)==0,\ T^2\,ca_{1,1}+T\,ca_{1,2}+T\,ca_{1,4}+T^2\,ca_{1,5}+2\,ca_{1,6}+2\,ca_{1,9}+T\,ca_{1,10}+2\,ca_{1,12}+2\,T\,ca_{1,13}+2\,T^2\,ca_{1,14}+T^2\,cc_{1,1}+T\,cc_{1,2}+2\,cc_{1,3}==0,\ T\left(T\,ca_{1,2}+T\,ca_{1,4}+4\,ca_{1,6}+4\,ca_{1,9}+T\,ca_{1,10}+4\,ca_{1,12}+2\,T\,ca_{1,13}+T\,cc_{1,2}+4\,cc_{1,3}\right)==0,\ 2\,T^2\left(ca_{1,6}+ca_{1,9}+ca_{1,12}+cc_{1,3}\right)==0\right\}$$

R1r

(Alt) In[•]:=
```
lhs = CF[Module[{es = {i, i⁺}},
    Times[
       Normal@Series[Exp[r_d[1, i, i⁺] + γ_d[-1, i⁺]], {ϵ, 0, d}],
       Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
      ] // Zip_{(p_#&/@es)∪(x_#&/@es)} // Expand
   ] //. {
    g_{iβ_} :> δ_{iβ} + T g_{i⁺,β} + (1 - T) g_{i⁺⁺,β}, g_{i⁺,β_} :> δ_{i⁺,β} + g_{i⁺⁺,β},
    g_{α_,i} :> T^-1 (g_{α,i⁺} - δ_{α,i⁺}), g_{α_,i⁺} :> T g_{α,i⁺⁺} - (1 - T) δ_{α,i⁺} - T δ_{α,i⁺⁺} }
 ]
```

(Alt) Out[•]=
$$1+\epsilon\left(ca_{1,1}-ca_{1,4}+ca_{1,5}-T\,ca_{1,5}+2\,ca_{1,12}-2\,ca_{1,13}+2\,T\,ca_{1,13}+2\,ca_{1,14}-4\,T\,ca_{1,14}+2\,T^2\,ca_{1,14}-cc_{1,1}-cc_{1,2}+T\,cc_{1,2}-2\,cc_{1,3}+4\,T\,cc_{1,3}-2\,T^2\,cc_{1,3}\right)+\epsilon\left(ca_{1,2}+T\,ca_{1,3}+ca_{1,4}+T\,ca_{1,5}-2\,ca_{1,9}+ca_{1,10}-2\,T\,ca_{1,10}+2\,T\,ca_{1,11}-2\,T^2\,ca_{1,11}-4\,ca_{1,12}+2\,ca_{1,13}-4\,T\,ca_{1,13}+4\,T\,ca_{1,14}-4\,T^2\,ca_{1,14}-T\,cc_{1,2}-4\,T\,cc_{1,3}+4\,T^2\,cc_{1,3}\right)g_{i^{++},i^{++}}+2\epsilon\left(ca_{1,6}+T\,ca_{1,7}+T^2\,ca_{1,8}+ca_{1,9}+T\,ca_{1,10}+T^2\,ca_{1,11}+ca_{1,12}+T\,ca_{1,13}+T^2\,ca_{1,14}-T^2\,cc_{1,3}\right)g^2_{i^{++},i^{++}}$$

(Alt) In[•]:=
```
1 + ϵ (ca_{1,1} + 2 ca_{1,8} - 2 ca_{1,9} + 2 T ca_{1,9} +
     2 ca_{1,10} - 4 T ca_{1,10} + 2 T^2 ca_{1,10} - cc_{1,1} - 2 cc_{1,2} + 4 T cc_{1,2} - 2 T^2 cc_{1,2}) +
  ϵ (-2 ca_{1,5} + ca_{1,6} - 2 T ca_{1,6} + 2 T ca_{1,7} - 2 T^2 ca_{1,7} - 4 ca_{1,8} + 2 ca_{1,9} - 4 T ca_{1,9} +
     4 T ca_{1,10} - 4 T^2 ca_{1,10} - 4 T cc_{1,2} + 4 T^2 cc_{1,2}) g_{i"++",i"++"} +
  2 ϵ (ca_{1,2} + T ca_{1,3} + T^2 ca_{1,4} + ca_{1,5} + T ca_{1,6} + T^2 ca_{1,7} + ca_{1,8} + T ca_{1,9} + T^2 ca_{1,10} - T^2 cc_{1,2}) g^2_{i"++",i"++"}
```

(Alt) Out[•]=
$$1+\epsilon\left(ca_{1,1}+2\,ca_{1,8}-2\,ca_{1,9}+2\,T\,ca_{1,9}+2\,ca_{1,10}-4\,T\,ca_{1,10}+2\,T^2\,ca_{1,10}-cc_{1,1}-2\,cc_{1,2}+4\,T\,cc_{1,2}-2\,T^2\,cc_{1,2}\right)+\epsilon\left(-2\,ca_{1,5}+ca_{1,6}-2\,T\,ca_{1,6}+2\,T\,ca_{1,7}-2\,T^2\,ca_{1,7}-4\,ca_{1,8}+2\,ca_{1,9}-4\,T\,ca_{1,9}+4\,T\,ca_{1,10}-4\,T^2\,ca_{1,10}-4\,T\,cc_{1,2}+4\,T^2\,cc_{1,2}\right)g_{i^{++},i^{++}}+2\epsilon\left(ca_{1,2}+T\,ca_{1,3}+T^2\,ca_{1,4}+ca_{1,5}+T\,ca_{1,6}+T^2\,ca_{1,7}+ca_{1,8}+T\,ca_{1,9}+T^2\,ca_{1,10}-T^2\,cc_{1,2}\right)g^2_{i^{++},i^{++}}$$

(Alt) In[•]:=
```
rhs = 1
```

(Alt) Out[•]=
$$1$$

(Alt) In[•]:=
```
me = Exponent[lhs - rhs, T, Min]
```

(Alt) Out[•]=
$$0$$

(Alt) In[•]:=
```
covars = DeleteCases[Variables[lhs - rhs], T | (ca | cb | cc | cd)__]
```

(Alt) Out[•]=
$$\{\epsilon,\ g_{i^{++},i^{++}}\}$$

(Alt) In[•]:=
```
eqnsR1r =
 (Factor[#] == 0) & /@ Union[Last /@ CoefficientRules[Expand[T^-me (lhs - rhs)], covars]]
```

(Alt) Out[•]=
$$\{2\,(ca_{1,6}+T\,ca_{1,7}+T^2\,ca_{1,8}+ca_{1,9}+T\,ca_{1,10}+T^2\,ca_{1,11}+ca_{1,12}+T\,ca_{1,13}+T^2\,ca_{1,14}-T^2\,cc_{1,3})==0,$$
$$ca_{1,1}-ca_{1,4}+ca_{1,5}-T\,ca_{1,5}+2\,ca_{1,12}-2\,ca_{1,13}+2\,T\,ca_{1,13}+2\,ca_{1,14}-$$
$$4\,T\,ca_{1,14}+2\,T^2\,ca_{1,14}-cc_{1,1}-cc_{1,2}+T\,cc_{1,2}-2\,cc_{1,3}+4\,T\,cc_{1,3}-2\,T^2\,cc_{1,3}==0,$$
$$ca_{1,2}+T\,ca_{1,3}+ca_{1,4}+T\,ca_{1,5}-2\,ca_{1,9}+ca_{1,10}-2\,T\,ca_{1,10}+2\,T\,ca_{1,11}-2\,T^2\,ca_{1,11}-$$
$$4\,ca_{1,12}+2\,ca_{1,13}-4\,T\,ca_{1,13}+4\,T\,ca_{1,14}-4\,T^2\,ca_{1,14}-T\,cc_{1,2}-4\,T\,cc_{1,3}+4\,T^2\,cc_{1,3}==0\}$$

Sw^+

(Alt) In[•]:=
```
lhs = CF[Module[{es = {i, j, i^+, j^+}},
    Times[
       Normal@
        Series[Exp[r_d[1, i, j] + ϒ_d[-1, i] + ϒ_d[-1, j] + ϒ_d[1, i^+] + ϒ_d[1, j^+]], {ϵ, 0, d}],
       Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
      ] // Zip[(p_#&/@es)∪(x_#&/@es)] // Expand
   ] //. gRules_{1,i,j}
 ]
```

(Alt) Out[•]=
$$1+\epsilon\,ca_{1,1}+\epsilon\,(ca_{1,2}-ca_{1,3}+T\,ca_{1,3})\,g_{i^+,i^+}+$$
$$2\,\epsilon\left(ca_{1,6}-ca_{1,7}+T\,ca_{1,7}+ca_{1,8}-2\,T\,ca_{1,8}+T^2\,ca_{1,8}\right)g^2_{i^+,i^+}+T\,\epsilon\,ca_{1,3}\,g_{i^+,j^+}+$$
$$2\,T\,\epsilon\,(ca_{1,7}-2\,ca_{1,8}+2\,T\,ca_{1,8})\,g_{i^+,i^+}\,g_{i^+,j^+}+2\,T^2\,\epsilon\,ca_{1,8}\,g^2_{i^+,j^+}-$$
$$\frac{\epsilon\left(-ca_{1,2}+T\,ca_{1,2}+ca_{1,3}-2\,T\,ca_{1,3}+T^2\,ca_{1,3}-ca_{1,4}+ca_{1,5}-T\,ca_{1,5}\right)g_{j^+,i^+}}{T}-$$
$$\frac{1}{T}\,2\,\epsilon\left(-2\,ca_{1,6}+2\,T\,ca_{1,6}+2\,ca_{1,7}-4\,T\,ca_{1,7}+2\,T^2\,ca_{1,7}-2\,ca_{1,8}+6\,T\,ca_{1,8}-6\,T^2\,ca_{1,8}+2\,T^3\,ca_{1,8}-\right.$$
$$\left.ca_{1,9}+ca_{1,10}-T\,ca_{1,10}-ca_{1,11}+2\,T\,ca_{1,11}-T^2\,ca_{1,11}+2\,cc_{1,3}-2\,T\,cc_{1,3}\right)g_{i^+,i^+}\,g_{j^+,i^+}+$$
$$\epsilon\left(2\,ca_{1,7}-2\,T\,ca_{1,7}-4\,ca_{1,8}+8\,T\,ca_{1,8}-4\,T^2\,ca_{1,8}+ca_{1,10}-2\,ca_{1,11}+2\,T\,ca_{1,11}\right)g_{i^+,j^+}\,g_{j^+,i^+}+$$
$$\frac{1}{T^2}\,2\,\epsilon\left(ca_{1,6}-2\,T\,ca_{1,6}+T^2\,ca_{1,6}-ca_{1,7}+3\,T\,ca_{1,7}-3\,T^2\,ca_{1,7}+T^3\,ca_{1,7}+ca_{1,8}-4\,T\,ca_{1,8}+\right.$$
$$6\,T^2\,ca_{1,8}-4\,T^3\,ca_{1,8}+T^4\,ca_{1,8}+ca_{1,9}-T\,ca_{1,9}-ca_{1,10}+2\,T\,ca_{1,10}-T^2\,ca_{1,10}+ca_{1,11}-$$
$$3\,T\,ca_{1,11}+3\,T^2\,ca_{1,11}-T^3\,ca_{1,11}+ca_{1,12}-ca_{1,13}+T\,ca_{1,13}+ca_{1,14}-2\,T\,ca_{1,14}+$$
$$\left.T^2\,ca_{1,14}-2\,cc_{1,3}+4\,T\,cc_{1,3}-2\,T^2\,cc_{1,3}\right)g^2_{j^+,i^+}+\epsilon\,(ca_{1,3}-T\,ca_{1,3}+ca_{1,5})\,g_{j^+,j^+}+$$
$$\epsilon\left(2\,ca_{1,7}-2\,T\,ca_{1,7}-4\,ca_{1,8}+8\,T\,ca_{1,8}-4\,T^2\,ca_{1,8}+ca_{1,10}-2\,ca_{1,11}+2\,T\,ca_{1,11}\right)g_{i^+,i^+}\,g_{j^+,j^+}-$$
$$2\,T\,\epsilon\,(-2\,ca_{1,8}+2\,T\,ca_{1,8}-ca_{1,11})\,g_{i^+,j^+}\,g_{j^+,j^+}+$$
$$\frac{1}{T}\,2\,\epsilon\left(ca_{1,7}-2\,T\,ca_{1,7}+T^2\,ca_{1,7}-2\,ca_{1,8}+6\,T\,ca_{1,8}-6\,T^2\,ca_{1,8}+2\,T^3\,ca_{1,8}+ca_{1,10}-T\,ca_{1,10}-\right.$$
$$\left.2\,ca_{1,11}+4\,T\,ca_{1,11}-2\,T^2\,ca_{1,11}+ca_{1,13}-2\,ca_{1,14}+2\,T\,ca_{1,14}+2\,cc_{1,3}-2\,T\,cc_{1,3}\right)g_{j^+,i^+}\,g_{j^+,j^+}+$$
$$2\,\epsilon\left(ca_{1,8}-2\,T\,ca_{1,8}+T^2\,ca_{1,8}+ca_{1,11}-T\,ca_{1,11}+ca_{1,14}\right)g^2_{j^+,j^+}$$

(Alt) In[ ]:=

```
rhs = CF[Module[{es = {i, j, i⁺, j⁺}},
    Times[
       Normal@Series[Exp[r_d[1, i, j]], {ϵ, 0, d}],
       Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
      ] // Zip_{(p_#&/@es)∪(x_#&/@es)} // Expand
   ] //. gRules_{1,i,j}
 ]
```

(Alt) Out[ ]=

$$1+\epsilon\, ca_{1,1}+\epsilon\,(ca_{1,2}-ca_{1,3}+T\,ca_{1,3})\,g_{i^+,i^+}+2\,\epsilon\left(ca_{1,6}-ca_{1,7}+T\,ca_{1,7}+ca_{1,8}-2\,T\,ca_{1,8}+T^2\,ca_{1,8}\right)g^2_{i^+,i^+}+T\,\epsilon\,ca_{1,3}\,g_{i^+,j^+}+2\,T\,\epsilon\,(ca_{1,7}-2\,ca_{1,8}+2\,T\,ca_{1,8})\,g_{i^+,i^+}\,g_{i^+,j^+}+2\,T^2\,\epsilon\,ca_{1,8}\,g^2_{i^+,j^+}-\frac{\epsilon\left(-ca_{1,2}+T\,ca_{1,2}+ca_{1,3}-2\,T\,ca_{1,3}+T^2\,ca_{1,3}-ca_{1,4}+ca_{1,5}-T\,ca_{1,5}\right)g_{j^+,i^+}}{T}-\frac{1}{T}2\,\epsilon\left(-2\,ca_{1,6}+2\,T\,ca_{1,6}+2\,ca_{1,7}-4\,T\,ca_{1,7}+2\,T^2\,ca_{1,7}-2\,ca_{1,8}+6\,T\,ca_{1,8}-6\,T^2\,ca_{1,8}+2\,T^3\,ca_{1,8}-ca_{1,9}+ca_{1,10}-T\,ca_{1,10}-ca_{1,11}+2\,T\,ca_{1,11}-T^2\,ca_{1,11}\right)g_{i^+,i^+}\,g_{j^+,i^+}+\epsilon\left(2\,ca_{1,7}-2\,T\,ca_{1,7}-4\,ca_{1,8}+8\,T\,ca_{1,8}-4\,T^2\,ca_{1,8}+ca_{1,10}-2\,ca_{1,11}+2\,T\,ca_{1,11}\right)g_{i^+,j^+}\,g_{j^+,i^+}+\frac{1}{T^2}2\,\epsilon\left(ca_{1,6}-2\,T\,ca_{1,6}+T^2\,ca_{1,6}-ca_{1,7}+3\,T\,ca_{1,7}-3\,T^2\,ca_{1,7}+T^3\,ca_{1,7}+ca_{1,8}-4\,T\,ca_{1,8}+6\,T^2\,ca_{1,8}-4\,T^3\,ca_{1,8}+T^4\,ca_{1,8}+ca_{1,9}-T\,ca_{1,9}-ca_{1,10}+2\,T\,ca_{1,10}-T^2\,ca_{1,10}+ca_{1,11}-3\,T\,ca_{1,11}+3\,T^2\,ca_{1,11}-T^3\,ca_{1,11}+ca_{1,12}-ca_{1,13}+T\,ca_{1,13}+ca_{1,14}-2\,T\,ca_{1,14}+T^2\,ca_{1,14}\right)g^2_{j^+,i^+}+\epsilon\,(ca_{1,3}-T\,ca_{1,3}+ca_{1,5})\,g_{j^+,j^+}+\epsilon\left(2\,ca_{1,7}-2\,T\,ca_{1,7}-4\,ca_{1,8}+8\,T\,ca_{1,8}-4\,T^2\,ca_{1,8}+ca_{1,10}-2\,ca_{1,11}+2\,T\,ca_{1,11}\right)g_{i^+,i^+}\,g_{j^+,j^+}-2\,T\,\epsilon\,(-2\,ca_{1,8}+2\,T\,ca_{1,8}-ca_{1,11})\,g_{i^+,j^+}\,g_{j^+,j^+}+\frac{1}{T}2\,\epsilon\left(ca_{1,7}-2\,T\,ca_{1,7}+T^2\,ca_{1,7}-2\,ca_{1,8}+6\,T\,ca_{1,8}-6\,T^2\,ca_{1,8}+2\,T^3\,ca_{1,8}+ca_{1,10}-T\,ca_{1,10}-2\,ca_{1,11}+4\,T\,ca_{1,11}-2\,T^2\,ca_{1,11}+ca_{1,13}-2\,ca_{1,14}+2\,T\,ca_{1,14}\right)g_{j^+,i^+}\,g_{j^+,j^+}+2\,\epsilon\left(ca_{1,8}-2\,T\,ca_{1,8}+T^2\,ca_{1,8}+ca_{1,11}-T\,ca_{1,11}+ca_{1,14}\right)g^2_{j^+,j^+}$$

(Alt) In[ ]:=

```
me = Exponent[lhs - rhs, T, Min]
```

(Alt) Out[ ]=

$$-2$$

(Alt) In[ ]:=

```
covars = DeleteCases[Variables[lhs - rhs], T | (ca | cb | cc | cd)__]
```

(Alt) Out[ ]=

$$\{\epsilon,\ g_{i^+,i^+},\ g_{j^+,i^+},\ g_{j^+,j^+}\}$$

(Alt) In[ ]:=

```
eqnsSwp = {}
```

(Alt) Out[ ]=

$$\{\}$$

Solution

(Alt) In[•]:=

```
vars = Cases[Variables[r_d[1, i1, j1] + r_d[-1, i2, j2] + ϒ_d[1, k]], (ca | cb | cc | cd)__]
{sol} = Solve[eqnsR3 ⋃ eqnsR2b ⋃ eqnsR2c ⋃ eqnsR1l ⋃ eqnsR1r ⋃ eqnsSwp, vars]
sol /. Rule → Set;
r_d[1, i, j]
r_d[-1, i, j]
ϒ_d[1, k]
ϒ_d[-1, k]
```

(Alt) Out[•]=

$$\{ca_{1,1}, ca_{1,2}, ca_{1,3}, ca_{1,4}, ca_{1,5}, ca_{1,6}, ca_{1,7}, ca_{1,8}, ca_{1,9}, ca_{1,10}, ca_{1,11}, ca_{1,12}, ca_{1,13}, ca_{1,14}, cb_{1,1}, cb_{1,2}, cb_{1,3}, cb_{1,4}, cb_{1,5}, cb_{1,6}, cb_{1,7}, cb_{1,8}, cb_{1,9}, cb_{1,10}, cb_{1,11}, cb_{1,12}, cb_{1,13}, cb_{1,14}, cc_{1,1}, cc_{1,2}, cc_{1,3}\}$$

Solve: Equations may not give solutions for all "solve" variables.

(Alt) Out[•]=

$$\left\{\left\{ca_{1,1} \to -\frac{ca_{1,2}}{2}, ca_{1,3} \to 0, ca_{1,4} \to -ca_{1,2}, ca_{1,5} \to 0, ca_{1,6} \to 0, ca_{1,7} \to 0, ca_{1,8} \to 0, ca_{1,9} \to -\frac{1}{2}(-1+T)\, ca_{1,10}, ca_{1,11} \to 0, ca_{1,12} \to -\frac{1}{2}(1-T)\, ca_{1,10}, ca_{1,13} \to -ca_{1,10}, ca_{1,14} \to 0, cb_{1,1} \to \frac{ca_{1,2}}{2}, cb_{1,2} \to -ca_{1,2}, cb_{1,3} \to 0, cb_{1,4} \to ca_{1,2}, cb_{1,5} \to 0, cb_{1,6} \to 0, cb_{1,7} \to 0, cb_{1,8} \to 0, cb_{1,9} \to -\frac{(-1+T)\, ca_{1,10}}{2T}, cb_{1,10} \to -ca_{1,10}, cb_{1,11} \to 0, cb_{1,12} \to -\frac{(1-T)\, ca_{1,10}}{2T}, cb_{1,13} \to ca_{1,10}, cb_{1,14} \to 0, cc_{1,1} \to \frac{ca_{1,2}}{2}, cc_{1,2} \to ca_{1,10}, cc_{1,3} \to 0\right\}\right\}$$

(Alt) Out[•]=

$$-\frac{1}{2}\in ca_{1,2} + \in p_i\, x_i\, ca_{1,2} - \in p_j\, x_i\, ca_{1,2} + \frac{1}{2}\in p_i\, p_j\, x_i^2\, ca_{1,10} - \frac{1}{2}T \in p_i\, p_j\, x_i^2\, ca_{1,10} - \frac{1}{2}\in p_j^2\, x_i^2\, ca_{1,10} + \frac{1}{2}T \in p_j^2\, x_i^2\, ca_{1,10} + \in p_i\, p_j\, x_i\, x_j\, ca_{1,10} - \in p_j^2\, x_i\, x_j\, ca_{1,10}$$

(Alt) Out[•]=

$$\frac{1}{2}\in ca_{1,2} - \in p_i\, x_i\, ca_{1,2} + \in p_j\, x_i\, ca_{1,2} - \frac{1}{2}\in p_i\, p_j\, x_i^2\, ca_{1,10} + \frac{\in p_i\, p_j\, x_i^2\, ca_{1,10}}{2T} + \frac{1}{2}\in p_j^2\, x_i^2\, ca_{1,10} - \frac{\in p_j^2\, x_i^2\, ca_{1,10}}{2T} - \in p_i\, p_j\, x_i\, x_j\, ca_{1,10} + \in p_j^2\, x_i\, x_j\, ca_{1,10}$$

(Alt) Out[•]=

$$\frac{1}{2}\in ca_{1,2} + \in p_k\, x_k\, ca_{1,10}$$

(Alt) Out[•]=

$$-\frac{1}{2}\in ca_{1,2} - \in p_k\, x_k\, ca_{1,10}$$

(Alt) In[•]:=

```
ca1,10 = 0;
rd[1, i, j]
rd[-1, i, j]
ϒd[1, k]
ϒd[-1, k]
```

(Alt) Out[•]=

$$-\frac{1}{2}\in \text{ca}_{1,2}+\in \text{p}_i\,\text{x}_i\,\text{ca}_{1,2}-\in \text{p}_j\,\text{x}_i\,\text{ca}_{1,2}$$

(Alt) Out[•]=

$$\frac{1}{2}\in \text{ca}_{1,2}-\in \text{p}_i\,\text{x}_i\,\text{ca}_{1,2}+\in \text{p}_j\,\text{x}_i\,\text{ca}_{1,2}$$

(Alt) Out[•]=

$$\frac{1}{2}\in \text{ca}_{1,2}$$

(Alt) Out[•]=

$$-\frac{1}{2}\in \text{ca}_{1,2}$$

(Alt) In[•]:=

```
lhs = CF[(r1[1, i, j] // px2g) //. gRules1,i,j /. {ϵ → 1, ca1,2 → 1, ca1,10 → -1}]
rhs = CF[R1[1, i, j] //. gRules1,i,j]
Simplify[lhs == rhs]
```

(Alt) Out[•]=

$$-\frac{1}{2}+\text{g}_{\text{i}^+,\text{i}^+}-\text{g}_{\text{j}^+,\text{i}^+}-\frac{(-1+\text{T})\,\text{g}_{\text{i}^+,\text{i}^+}\,\text{g}_{\text{j}^+,\text{i}^+}}{\text{T}}-\text{g}_{\text{i}^+,\text{j}^+}\,\text{g}_{\text{j}^+,\text{i}^+}+\frac{(-1+\text{T})\,\text{g}_{\text{j}^+,\text{i}^+}^2}{\text{T}}-\text{g}_{\text{i}^+,\text{i}^+}\,\text{g}_{\text{j}^+,\text{j}^+}+2\,\text{g}_{\text{j}^+,\text{i}^+}\,\text{g}_{\text{j}^+,\text{j}^+}$$

(Alt) Out[•]=

$$-\frac{1}{2}+\text{g}_{\text{i}^+,\text{i}^+}-\text{g}_{\text{j}^+,\text{i}^+}-\frac{(-1+\text{T})\,\text{g}_{\text{i}^+,\text{i}^+}\,\text{g}_{\text{j}^+,\text{i}^+}}{\text{T}}-\text{g}_{\text{i}^+,\text{j}^+}\,\text{g}_{\text{j}^+,\text{i}^+}+\frac{(-1+\text{T})\,\text{g}_{\text{j}^+,\text{i}^+}^2}{\text{T}}-\text{g}_{\text{i}^+,\text{i}^+}\,\text{g}_{\text{j}^+,\text{j}^+}+2\,\text{g}_{\text{j}^+,\text{i}^+}\,\text{g}_{\text{j}^+,\text{j}^+}$$

(Alt) Out[•]=

```
True
```