Pensieve header: Implementing ρ_1, and also ρ_d.

exec

```
nb2tex$TeXFileName = "Rho1.tex";
```

pdf

Preliminaries

pdf

This is Rho.nb of http://drorbn.net/oa22/ap.

(Alt) In[ ]:=

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Oaxaca-2210"];
```

(Alt) In[ ]:=
pdf

```
Once[<< KnotTheory`; << Rot.m];
```

pdf

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/la22/ap to compute rotation numbers.
```

pdf

The Program

(Alt) In[ ]:=
pdf

```
R1[s_, i_, j_] := s (gji (gj+,j + gj,j+ - gij) - gii (gj,j+ - 1) - 1 / 2);
Z[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ((-T^s  T^s - 1
                                                               0     -1 )))];
   Δ = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
   G = Inverse[A];
   ρ1 = Sum_{k=1}^{n} R1 @@ Cs[[k]] - Sum_{k=1}^{2 n} φ[[k]] (gkk - 1 / 2);
   Factor@{Δ, Δ^2 ρ1 /. α_+ :> α + 1 /. gα_,β_ :> G[[α, β]]}];
```

pdf

The First Few Knots

pdf

In[•]:=
```
TableForm[Table[Join[{K[[1]]_K[[2]]}, Z[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]//TableForm=

pdf

3_1	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2\,(1+T^2)}{T^2}$
4_1	$-\frac{1-3\,T+T^2}{T}$	0
5_1	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2\,(1+T^2)\,(2+T^2+2\,T^4)}{T^4}$
5_2	$\frac{2-3\,T+2\,T^2}{T}$	$\frac{(-1+T)^2\,(5-4\,T+5\,T^2)}{T^2}$
6_1	$-\frac{(-2+T)\,(-1+2\,T)}{T}$	$\frac{(-1+T)^2\,(1-4\,T+T^2)}{T^2}$
6_2	$-\frac{1-3\,T+3\,T^2-3\,T^3+T^4}{T^2}$	$\frac{(-1+T)^2\,(1-4\,T+4\,T^2-4\,T^3+4\,T^4-4\,T^5+T^6)}{T^4}$
6_3	$\frac{1-3\,T+5\,T^2-3\,T^3+T^4}{T^2}$	0

tex

```
\def\nbpdfText#1{\vskip -3mm\[ \includegraphics[width=0.4\linewidth]{#1}\quad p=1-T^s \]}
```

pdf

tex

```
\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics{#1}}
```

tex

```
\needspace{2in}
```

pdf

Fast!

tex

```
\[ \resizebox{\linewidth}{!}{\import{../Waco-2203/}{GST48-Marked.pdf_t}} \]
```

pdf

In[•]:=
```
Timing@
 Z[GST48 = EPD[X14,1, X2,29, X3,40, X43,4, X26,5, X6,95, X96,7, X13,8, X9,28, X10,41, X42,11, X27,12, X30,15,
    X16,61, X17,72, X18,83, X19,34, X89,20, X21,92, X79,22, X68,23, X57,24, X25,56, X62,31, X73,32,
    X84,33, X50,35, X36,81, X37,70, X38,59, X39,54, X44,55, X58,45, X69,46, X80,47, X48,91, X90,49,
    X51,82, X52,71, X53,60, X63,74, X64,85, X76,65, X87,66, X67,94, X75,86, X88,77, X78,93]]
```

Out[•]=

pdf

$$\left\{170.313,\ \left\{-\frac{\left(-1+2\,T-T^2-T^3+2\,T^4-T^5+T^8\right)\,\left(-1+T^3-2\,T^4+T^5+T^6-2\,T^7+T^8\right)}{T^8},\right.\right.$$
$$\frac{1}{T^{16}}\,(-1+T)^2\,\left(5-18\,T+33\,T^2-32\,T^3+2\,T^4+42\,T^5-62\,T^6-8\,T^7+166\,T^8-242\,T^9+108\,T^{10}+\right.$$
$$\left.\left.\left.132\,T^{11}-226\,T^{12}+148\,T^{13}-11\,T^{14}-36\,T^{15}-11\,T^{16}+148\,T^{17}-226\,T^{18}+132\,T^{19}+108\,T^{20}-\right.\right.\right.$$
$$\left.\left.\left.242\,T^{21}+166\,T^{22}-8\,T^{23}-62\,T^{24}+42\,T^{25}+2\,T^{26}-32\,T^{27}+33\,T^{28}-18\,T^{29}+5\,T^{30}\right)\right\}\right\}$$

pdf

Strong!

pdf

```
{NumberOfKnots[{3, 12}],
 Length@Union@Table[Z[K], {K, AllKnots[{3, 12}]}],
 Length@Union@Table[{HOMFLYPT[K], Kh[K]}, {K, AllKnots[{3, 12}]}]}
```

Out[•]=

pdf

{2977, 2882, 2785}

In[•]:=
```
2977 - {2882, 2785}
```

Out[•]=

{95, 192}

tex

So the pair (Δ,ρ_1) attains 2,882 distinct values on the 2,977 prime knots with up to 12 crossings (a deficit of 95), whereas the pair (HOMFLYPT, Khovanov Homology) attains only 2,785 distinct values on the same knots (a deficit of 192).

tex

\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics[width=\linewidth]{#1}}

pdf

tex

\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics{#1}}

Invariance under R3

exec

```
nb2tex$TeXFileName = "Invariance.tex";
```

pdf

```
In[•]:= δ_i_,j_ := If[i === j, 1, 0];
gRules_s_,i_,j_ := {g_iβ_ :> δ_iβ + T^s g_i⁺,β + (1 - T^s) g_j⁺,β, g_jβ_ :> δ_jβ + g_j⁺,β,
  g_α_,i :> T^-s (g_α,i⁺ - δ_α,i⁺), g_α_j :> g_α,j⁺ - (1 - T^s) g_αi - δ_α,j⁺}
```

Proof of Reidemeister 3:

pdf

```
In[•]:= lhs = R1[1, j, k] + R1[1, i, k⁺] + R1[1, i⁺, j⁺] //. gRules1,j,k ⋃ gRules1,i,k⁺ ⋃ gRules1,i⁺,j⁺;
rhs = R1[1, i, j] + R1[1, i⁺, k] + R1[1, j⁺, k⁺] //. gRules1,i,j ⋃ gRules1,i⁺,k ⋃ gRules1,j⁺,k⁺;
Simplify[lhs == rhs]
```

Out[•]=
pdf

```
True
```

tex

Next comes Reid1, where we use results from an earlier example:

```
In[•]:= {{1, T^-1, 1}, {0, T^-1, 1}, {0, 0, 1}} // Inverse // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & T & -T \\ 0 & 0 & 1 \end{pmatrix}$$

pdf

```
In[•]:= R1[1, 2, 1] - 1 (g22 - 1 / 2) /. g_α_,β_ :> {{1, T^-1, 1}, {0, T^-1, 1}, {0, 0, 1}}[[α, β]]
```

Out[•]=
pdf

$$\frac{1}{T^2} - \frac{1}{T} - \frac{-1 + \frac{1}{T}}{T}$$

tex

Invariance under the other moves is proven similarly.

exec

```
nb2tex$TeXFileName = "Rhod.tex";
nb2tex$PDFWidth = 4.2 / 0.7;
```

On to ρ_d!

tex

{\bf\red Implementation.} Data, then program (with output using the \text{Conway} variable $z=\sqrt{T}-1/\sqrt{T}$), and then a demo. See {\tt Rho.nb} of \web{ap}.

\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.7]{#1}}
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.7]{#1}}

(Alt) In[ ]:= pdf

```
V@γ1,φ_[k_] = φ (1 / 2 - p̄k x̄k); V@γ2,φ_[k_] = -φ^2 p̄k x̄k / 2; V@γ3,φ_[k_] := -φ^3 p̄k x̄k / 6
```

(Alt) In[ ]:= pdf

```
V@r1,s_[i_, j_] :=
 s (-1 + 2 pi xi - 2 pj xi + (-1 + T^s) pi pj xi^2 + (1 - T^s) pj^2 xi^2 - 2 pi pj xi xj + 2 pj^2 xi xj) / 2
```

(Alt) In[ ]:= pdf

```
V@r2,1[i_, j_] := (-6 pi xi + 6 pj xi - 3 (-1 + 3 T) pi pj xi^2 + 3 (-1 + 3 T) pj^2 xi^2 + 4 (-1 + T) pi^2 pj xi^3 -
    2 (-1 + T) (5 + T) pi pj^2 xi^3 + 2 (-1 + T) (3 + T) pj^3 xi^3 + 18 pi pj xi xj - 18 pj^2 xi xj -
    6 pi^2 pj xi^2 xj + 6 (2 + T) pi pj^2 xi^2 xj - 6 (1 + T) pj^3 xi^2 xj - 6 pi pj^2 xi xj^2 + 6 pj^3 xi xj^2) / 12
```

(Alt) In[ ]:= pdf

```
V@r2,-1[i_, j_] :=
 (-6 T^2 pi xi + 6 T^2 pj xi + 3 (-3 + T) T pi pj xi^2 - 3 (-3 + T) T pj^2 xi^2 - 4 (-1 + T) T pi^2 pj xi^3 + 2 (-1 + T)
     (1 + 5 T) pi pj^2 xi^3 - 2 (-1 + T) (1 + 3 T) pj^3 xi^3 + 18 T^2 pi pj xi xj - 18 T^2 pj^2 xi xj - 6 T^2 pi^2 pj xi^2 xj +
   6 T (1 + 2 T) pi pj^2 xi^2 xj - 6 T (1 + T) pj^3 xi^2 xj - 6 T^2 pi pj^2 xi xj^2 + 6 T^2 pj^3 xi xj^2) / (12 T^2)
```

(Alt) In[ ]:= pdf

```
V@r3,1[i_, j_] :=
 (4 pi xi - 4 pj xi + 2 (5 + 7 T) pi pj xi^2 - 2 (5 + 7 T) pj^2 xi^2 - 4 (-5 + 6 T) pi^2 pj xi^3 + 4 (-16 + 17 T + 2 T^2)
     pi pj^2 xi^3 - 4 (-11 + 11 T + 2 T^2) pj^3 xi^3 + 3 (-1 + T) pi^3 pj xi^4 - 3 (-1 + T) (4 + 3 T) pi^2 pj^2 xi^4 +
   (-1 + T) (13 + 22 T + T^2) pi pj^3 xi^4 - (-1 + T) (4 + 13 T + T^2) pj^4 xi^4 - 28 pi pj xi xj + 28 pj^2 xi xj +
   36 pi^2 pj xi^2 xj - 12 (9 + 2 T) pi pj^2 xi^2 xj + 24 (3 + T) pj^3 xi^2 xj - 4 pi^3 pj xi^3 xj + 28 T pi^2 pj^2 xi^3 xj -
   4 (-6 + 17 T + T^2) pi pj^3 xi^3 xj + 4 (-5 + 10 T + T^2) pj^4 xi^3 xj + 24 pi pj^2 xi xj^2 - 24 pj^3 xi xj^2 -
   24 pi^2 pj^2 xi^2 xj^2 + 6 (10 + T) pi pj^3 xi^2 xj^2 - 6 (6 + T) pj^4 xi^2 xj^2 - 4 pi pj^3 xi xj^3 + 4 pj^4 xi xj^3) / 24
```

(Alt) In[ ]:= pdf

```
V@r3,-1[i_, j_] :=
 (-4 T^3 pi xi + 4 T^3 pj xi - 2 T^2 (7 + 5 T) pi pj xi^2 + 2 T^2 (7 + 5 T) pj^2 xi^2 - 4 T^2 (-6 + 5 T) pi^2 pj xi^3 +
   4 T (-2 - 17 T + 16 T^2) pi pj^2 xi^3 - 4 T (-2 - 11 T + 11 T^2) pj^3 xi^3 + 3 (-1 + T) T^2 pi^3 pj xi^4 -
   3 (-1 + T) T (3 + 4 T) pi^2 pj^2 xi^4 + (-1 + T) (1 + 22 T + 13 T^2) pi pj^3 xi^4 -
   (-1 + T) (1 + 13 T + 4 T^2) pj^4 xi^4 + 28 T^3 pi pj xi xj - 28 T^3 pj^2 xi xj - 36 T^3 pi^2 pj xi^2 xj +
   12 T^2 (2 + 9 T) pi pj^2 xi^2 xj - 24 T^2 (1 + 3 T) pj^3 xi^2 xj + 4 T^3 pi^3 pj xi^3 xj -
   28 T^2 pi^2 pj^2 xi^3 xj - 4 T (-1 - 17 T + 6 T^2) pi pj^3 xi^3 xj + 4 T (-1 - 10 T + 5 T^2) pj^4 xi^3 xj -
   24 T^3 pi pj^2 xi xj^2 + 24 T^3 pj^3 xi xj^2 + 24 T^3 pi^2 pj^2 xi^2 xj^2 - 6 T^2 (1 + 10 T) pi pj^3 xi^2 xj^2 +
   6 T^2 (1 + 6 T) pj^4 xi^2 xj^2 + 4 T^3 pi pj^3 xi xj^3 - 4 T^3 pj^4 xi xj^3) / (24 T^3)
```

(Alt) In[•]:=
pdf

```
{p*, x*, p̄*, x̄*} = {π, ξ, π̄, ξ̄};  (z_i__)* := (z*)_i;
Zip_{}[ℰ_] := ℰ;
Zip_{z_,zs___}[ℰ_] := (Collect[ℰ // Zip_{zs}, z] /. f_. z^d_· :> (D[f, {z*, d}])) /. z* → 0
```

(Alt) In[•]:=
pdf

```
gPair[fs_, w_] := gPair[fs, w] = Collect[Zip_{Join@@Table[{p_α, p̄_α, x_α, x̄_α}, {α, w}]}[(Times @@ (V /@ fs))
    Exp[Sum[g_{α,β} (π_α + π̄_α) (ξ_β + ξ̄_β), {α, w}, {β, w}] - Sum[ξ̄_α π_α, {α, w}]]], g__, Factor]
```

(Alt) In[•]:=
pdf

```
T2z[p_] := Module[{q = Expand[p], n, c},
  If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
   c z^(2 n) + T2z[q - c (T^(1/2) - T^(-1/2))^(2 n)]]];
```

(Alt) In[•]:=
pdf

```
Z_d_[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, d1, Z1, Z2, Z3},
  {Cs, φ} = Rot[K]; n = Length[Cs]; A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
  {Δ, G} = Factor@{T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det@A, Inverse@A};
  Z1 = Exp[Total[Cases[Cs, {s_, i_, j_} :> Sum[ϵ^d1 r_{d1,s}[i, j], {d1, d}]]] +
      Sum[ϵ^d1 γ_{d1,φ[[k]]}[k], {k, 2 n}, {d1, d}] /. γ_,0[_] → 0];
  Z2 = Expand[F[{}, {}] × Normal@Series[Z1, {ϵ, 0, d}]] //. F[fs_, {es___}] ×
      (f : (r | γ)_ps__[is__])^p_· :> F[Join[fs, Table[f, p]], DeleteDuplicates@{es, is}];
  Z3 = Expand[Z2 /. F[fs_, es_] :> Expand[gPair[
        Replace[fs, Thread[es → Range@Length@es], {2}], Length@es
      ] /. g_α_,β_ :> G[[es[[α]], es[[β]]]]]];
  Collect[{Δ, Z3 /. ϵ^p_· → p! Δ^(2 p) ϵ^p}, ϵ, T2z]];
```

(Alt) In[•]:=

```
K = Knot[4, 1]; d = 2;
```

(Alt) In[•]:=

```
{Cs, φ} = Rot[K]; n = Length[Cs]; A = IdentityMatrix[2 n + 1];
Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
{Δ, G} = Factor@{T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det@A, Inverse@A};
```

KnotTheory: Loading precomputed data in PD4Knots`.

(Alt) In[•]:=

```
Z1 = Exp[Total[Cases[Cs, {s_, i_, j_} :> Sum[ϵ^d1 r_{d1,s}[i, j], {d1, d}]]] +
     Sum[ϵ^d1 γ_{d1,φ[[k]]}[k], {k, 2 n}, {d1, d}] /. γ_,0[_] → 0]
```

(Alt) Out[•]=

$$\epsilon^{\epsilon\, r_{1,-1}[3,6]+\epsilon\, r_{1,-1}[7,2]+\epsilon\, r_{1,1}[1,4]+\epsilon\, r_{1,1}[5,8]+\epsilon^2 r_{2,-1}[3,6]+\epsilon^2 r_{2,-1}[7,2]+\epsilon^2 r_{2,1}[1,4]+\epsilon^2 r_{2,1}[5,8]+\epsilon\,\gamma_{1,-1}[4]+\epsilon\,\gamma_{1,-1}[7]+\epsilon^2\gamma_{2,-1}[4]+}$$

(Alt) In[•]:=

```
Z2 = Expand[F[{}, {}] × Normal@Series[Z1, {ϵ, 0, d}]] //.
  F[fs_, {es___}] × (f : (r | γ)_ps__[is__])^p_. :>
   F[Join[fs, Table[f, p]], DeleteDuplicates@{es, is}]
```

(Alt) Out[•]=

$$\begin{aligned}
&F[\{\},\{\}]+\epsilon\,F[\{r_{1,-1}[3,6]\},\{3,6\}]+\epsilon\,F[\{r_{1,-1}[7,2]\},\{7,2\}]+\\
&\epsilon\,F[\{r_{1,1}[1,4]\},\{1,4\}]+\epsilon\,F[\{r_{1,1}[5,8]\},\{5,8\}]+\epsilon^2 F[\{r_{2,-1}[3,6]\},\{3,6\}]+\\
&\epsilon^2 F[\{r_{2,-1}[7,2]\},\{7,2\}]+\epsilon^2 F[\{r_{2,1}[1,4]\},\{1,4\}]+\epsilon^2 F[\{r_{2,1}[5,8]\},\{5,8\}]+\\
&\epsilon\,F[\{\gamma_{1,-1}[4]\},\{4\}]+\epsilon\,F[\{\gamma_{1,-1}[7]\},\{7\}]+\epsilon^2 F[\{\gamma_{2,-1}[4]\},\{4\}]+\\
&\epsilon^2 F[\{\gamma_{2,-1}[7]\},\{7\}]+\frac{1}{2}\epsilon^2 F[\{r_{1,-1}[3,6],r_{1,-1}[3,6]\},\{3,6\}]+\\
&\epsilon^2 F[\{r_{1,-1}[3,6],r_{1,-1}[7,2]\},\{3,6,7,2\}]+\epsilon^2 F[\{r_{1,-1}[3,6],r_{1,1}[1,4]\},\{3,6,1,4\}]+\\
&\epsilon^2 F[\{r_{1,-1}[3,6],r_{1,1}[5,8]\},\{3,6,5,8\}]+\epsilon^2 F[\{r_{1,-1}[3,6],\gamma_{1,-1}[4]\},\{3,6,4\}]+\\
&\epsilon^2 F[\{r_{1,-1}[3,6],\gamma_{1,-1}[7]\},\{3,6,7\}]+\frac{1}{2}\epsilon^2 F[\{r_{1,-1}[7,2],r_{1,-1}[7,2]\},\{7,2\}]+\\
&\epsilon^2 F[\{r_{1,-1}[7,2],r_{1,1}[1,4]\},\{7,2,1,4\}]+\epsilon^2 F[\{r_{1,-1}[7,2],r_{1,1}[5,8]\},\{7,2,5,8\}]+\\
&\epsilon^2 F[\{r_{1,-1}[7,2],\gamma_{1,-1}[4]\},\{7,2,4\}]+\epsilon^2 F[\{r_{1,-1}[7,2],\gamma_{1,-1}[7]\},\{7,2\}]+\\
&\frac{1}{2}\epsilon^2 F[\{r_{1,1}[1,4],r_{1,1}[1,4]\},\{1,4\}]+\epsilon^2 F[\{r_{1,1}[1,4],r_{1,1}[5,8]\},\{1,4,5,8\}]+\\
&\epsilon^2 F[\{r_{1,1}[1,4],\gamma_{1,-1}[4]\},\{1,4\}]+\epsilon^2 F[\{r_{1,1}[1,4],\gamma_{1,-1}[7]\},\{1,4,7\}]+\\
&\frac{1}{2}\epsilon^2 F[\{r_{1,1}[5,8],r_{1,1}[5,8]\},\{5,8\}]+\epsilon^2 F[\{r_{1,1}[5,8],\gamma_{1,-1}[4]\},\{5,8,4\}]+\\
&\epsilon^2 F[\{r_{1,1}[5,8],\gamma_{1,-1}[7]\},\{5,8,7\}]+\frac{1}{2}\epsilon^2 F[\{\gamma_{1,-1}[4],\gamma_{1,-1}[4]\},\{4\}]+\\
&\epsilon^2 F[\{\gamma_{1,-1}[4],\gamma_{1,-1}[7]\},\{4,7\}]+\frac{1}{2}\epsilon^2 F[\{\gamma_{1,-1}[7],\gamma_{1,-1}[7]\},\{7\}]
\end{aligned}$$

(Alt) In[•]:=

```
Expand[Z2 /. F[fs_, es_] :> Expand[gPair0[
     Replace[fs, Thread[es → Range@Length@es], {2}], Length@es
    ]]]
```

(Alt) Out[•]=

$$\begin{aligned}
&gPair0[\{\},0]+2\epsilon\,gPair0[\{r_{1,-1}[1,2]\},2]+2\epsilon\,gPair0[\{r_{1,1}[1,2]\},2]+\\
&2\epsilon^2 gPair0[\{r_{2,-1}[1,2]\},2]+2\epsilon^2 gPair0[\{r_{2,1}[1,2]\},2]+2\epsilon\,gPair0[\{\gamma_{1,-1}[1]\},1]+\\
&2\epsilon^2 gPair0[\{\gamma_{2,-1}[1]\},1]+\epsilon^2 gPair0[\{r_{1,-1}[1,2],r_{1,-1}[1,2]\},2]+\\
&\epsilon^2 gPair0[\{r_{1,-1}[1,2],r_{1,-1}[3,4]\},4]+4\epsilon^2 gPair0[\{r_{1,-1}[1,2],r_{1,1}[3,4]\},4]+\\
&\epsilon^2 gPair0[\{r_{1,-1}[1,2],\gamma_{1,-1}[1]\},2]+3\epsilon^2 gPair0[\{r_{1,-1}[1,2],\gamma_{1,-1}[3]\},3]+\\
&\epsilon^2 gPair0[\{r_{1,1}[1,2],r_{1,1}[1,2]\},2]+\epsilon^2 gPair0[\{r_{1,1}[1,2],r_{1,1}[3,4]\},4]+\\
&\epsilon^2 gPair0[\{r_{1,1}[1,2],\gamma_{1,-1}[2]\},2]+3\epsilon^2 gPair0[\{r_{1,1}[1,2],\gamma_{1,-1}[3]\},3]+\\
&\epsilon^2 gPair0[\{\gamma_{1,-1}[1],\gamma_{1,-1}[1]\},1]+\epsilon^2 gPair0[\{\gamma_{1,-1}[1],\gamma_{1,-1}[2]\},2]
\end{aligned}$$

(Alt) In[•]:=

```
Z3 = Expand[Z2 /. F[fs_, es_] :> Expand[gPair[
      Replace[fs, Thread[es → Range@Length@es], {2}], Length@es
    ]]]
```

(Alt) Out[•]=

$$1-\epsilon+\frac{\epsilon^2}{2}+2\,\epsilon\,g_{1,1}-4\,\epsilon^2\,g_{1,1}+4\,\epsilon^2\,g_{1,1}^2+4\,\epsilon^2\,g_{2,1}-\frac{2\,\epsilon\,g_{1,1}\,g_{2,1}}{T}+2\,T\,\epsilon\,g_{1,1}\,g_{2,1}-3\,\epsilon^2\,g_{1,1}\,g_{2,1}+$$
$$\frac{\epsilon^2\,g_{1,1}\,g_{2,1}}{2\,T}-\frac{15}{2}\,T\,\epsilon^2\,g_{1,1}\,g_{2,1}-17\,\epsilon^2\,g_{1,1}^2\,g_{2,1}+\frac{7\,\epsilon^2\,g_{1,1}^2\,g_{2,1}}{T}+10\,T\,\epsilon^2\,g_{1,1}^2\,g_{2,1}+9\,\epsilon^2\,g_{1,2}\,g_{2,1}-$$
$$20\,\epsilon^2\,g_{1,1}\,g_{1,2}\,g_{2,1}+\frac{2\,\epsilon\,g_{2,1}^2}{T}-2\,T\,\epsilon\,g_{2,1}^2-\epsilon^2\,g_{2,1}^2+\frac{\epsilon^2\,g_{2,1}^2}{2\,T}+\frac{17}{2}\,T\,\epsilon^2\,g_{2,1}^2+41\,\epsilon^2\,g_{1,1}\,g_{2,1}^2-$$
$$\frac{2\,\epsilon^2\,g_{1,1}\,g_{2,1}^2}{T^2}-\frac{17\,\epsilon^2\,g_{1,1}\,g_{2,1}^2}{T}-20\,T\,\epsilon^2\,g_{1,1}\,g_{2,1}^2-2\,T^2\,\epsilon^2\,g_{1,1}\,g_{2,1}^2+12\,\epsilon^2\,g_{1,1}^2\,g_{2,1}^2+\frac{6\,\epsilon^2\,g_{1,1}^2\,g_{2,1}^2}{T^2}-$$
$$\frac{12\,\epsilon^2\,g_{1,1}^2\,g_{2,1}^2}{T}-12\,T\,\epsilon^2\,g_{1,1}^2\,g_{2,1}^2+6\,T^2\,\epsilon^2\,g_{1,1}^2\,g_{2,1}^2+21\,\epsilon^2\,g_{1,2}\,g_{2,1}^2+\frac{2\,\epsilon^2\,g_{1,2}\,g_{2,1}^2}{T}+3\,T\,\epsilon^2\,g_{1,2}\,g_{2,1}^2+$$
$$24\,\epsilon^2\,g_{1,1}\,g_{1,2}\,g_{2,1}^2-\frac{12\,\epsilon^2\,g_{1,1}\,g_{1,2}\,g_{2,1}^2}{T}-12\,T\,\epsilon^2\,g_{1,1}\,g_{1,2}\,g_{2,1}^2+8\,\epsilon^2\,g_{1,2}^2\,g_{2,1}^2-24\,\epsilon^2\,g_{2,1}^3+$$
$$\frac{2\,\epsilon^2\,g_{2,1}^3}{T^2}+\frac{10\,\epsilon^2\,g_{2,1}^3}{T}+10\,T\,\epsilon^2\,g_{2,1}^3+2\,T^2\,\epsilon^2\,g_{2,1}^3-24\,\epsilon^2\,g_{1,1}\,g_{2,1}^3-\frac{12\,\epsilon^2\,g_{1,1}\,g_{2,1}^3}{T^2}+\frac{24\,\epsilon^2\,g_{1,1}\,g_{2,1}^3}{T}+$$
$$24\,T\,\epsilon^2\,g_{1,1}\,g_{2,1}^3-12\,T^2\,\epsilon^2\,g_{1,1}\,g_{2,1}^3-24\,\epsilon^2\,g_{1,2}\,g_{2,1}^3+\frac{12\,\epsilon^2\,g_{1,2}\,g_{2,1}^3}{T}+12\,T\,\epsilon^2\,g_{1,2}\,g_{2,1}^3+12\,\epsilon^2\,g_{2,1}^4+$$
$$\frac{6\,\epsilon^2\,g_{2,1}^4}{T^2}-\frac{12\,\epsilon^2\,g_{2,1}^4}{T}-12\,T\,\epsilon^2\,g_{2,1}^4+6\,T^2\,\epsilon^2\,g_{2,1}^4-\epsilon^2\,g_{2,2}+9\,\epsilon^2\,g_{1,1}\,g_{2,2}-10\,\epsilon^2\,g_{1,1}^2\,g_{2,2}-$$
$$20\,\epsilon^2\,g_{2,1}\,g_{2,2}+42\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{2,2}+\frac{4\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{2,2}}{T}+6\,T\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{2,2}+24\,\epsilon^2\,g_{1,1}^2\,g_{2,1}\,g_{2,2}-$$
$$\frac{12\,\epsilon^2\,g_{1,1}^2\,g_{2,1}\,g_{2,2}}{T}-12\,T\,\epsilon^2\,g_{1,1}^2\,g_{2,1}\,g_{2,2}-12\,\epsilon^2\,g_{1,2}\,g_{2,1}\,g_{2,2}+32\,\epsilon^2\,g_{1,1}\,g_{1,2}\,g_{2,1}\,g_{2,2}-$$
$$33\,\epsilon^2\,g_{2,1}^2\,g_{2,2}-\frac{6\,\epsilon^2\,g_{2,1}^2\,g_{2,2}}{T}-9\,T\,\epsilon^2\,g_{2,1}^2\,g_{2,2}-72\,\epsilon^2\,g_{1,1}\,g_{2,1}^2\,g_{2,2}+\frac{36\,\epsilon^2\,g_{1,1}\,g_{2,1}^2\,g_{2,2}}{T}+$$
$$36\,T\,\epsilon^2\,g_{1,1}\,g_{2,1}^2\,g_{2,2}-48\,\epsilon^2\,g_{1,2}\,g_{2,1}^2\,g_{2,2}+48\,\epsilon^2\,g_{2,1}^3\,g_{2,2}-\frac{24\,\epsilon^2\,g_{2,1}^3\,g_{2,2}}{T}-24\,T\,\epsilon^2\,g_{2,1}^3\,g_{2,2}-$$
$$6\,\epsilon^2\,g_{1,1}\,g_{2,2}^2+8\,\epsilon^2\,g_{1,1}^2\,g_{2,2}^2+18\,\epsilon^2\,g_{2,1}\,g_{2,2}^2-48\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{2,2}^2+48\,\epsilon^2\,g_{2,1}^2\,g_{2,2}^2-2\,\epsilon^2\,g_{1,3}\,g_{3,1}+$$
$$2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,1}-\frac{6\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,1}}{T}+4\,T\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,1}+2\,\epsilon^2\,g_{1,3}\,g_{2,2}\,g_{3,1}+2\,\epsilon^2\,g_{2,3}\,g_{3,1}+$$
$$2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,1}-\frac{6\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,1}}{T}+4\,T\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,1}+2\,\epsilon^2\,g_{1,2}\,g_{2,3}\,g_{3,1}-4\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,1}+$$
$$\frac{12\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,1}}{T}-8\,T\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,1}-4\,\epsilon^2\,g_{2,2}\,g_{2,3}\,g_{3,1}+2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,2}+2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,2}-$$
$$4\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,2}+\epsilon^2\,g_{3,3}-2\,\epsilon^2\,g_{1,1}\,g_{3,3}+2\,\epsilon^2\,g_{2,1}\,g_{3,3}+2\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{3,3}-\frac{6\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{3,3}}{T}+$$

$$4\,T\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{3,3}+2\,\epsilon^2\,g_{1,2}\,g_{2,1}\,g_{3,3}-2\,\epsilon^2\,g_{2,1}^2\,g_{3,3}+\frac{6\,\epsilon^2\,g_{2,1}^2\,g_{3,3}}{T}-4\,T\,\epsilon^2\,g_{2,1}^2\,g_{3,3}+$$

$$2\,\epsilon^2\,g_{1,1}\,g_{2,2}\,g_{3,3}-4\,\epsilon^2\,g_{2,1}\,g_{2,2}\,g_{3,3}+2\,\epsilon^2\,g_{1,3}\,g_{4,1}-2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{4,1}+\frac{3\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{4,1}}{T}-$$

$$T\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{4,1}-2\,\epsilon^2\,g_{1,3}\,g_{2,2}\,g_{4,1}-2\,\epsilon^2\,g_{2,3}\,g_{4,1}-2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{4,1}+\frac{3\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{4,1}}{T}-$$

$$T\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{4,1}-2\,\epsilon^2\,g_{1,2}\,g_{2,3}\,g_{4,1}+4\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{4,1}-\frac{6\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{4,1}}{T}+2\,T\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{4,1}+$$

$$4\,\epsilon^2\,g_{2,2}\,g_{2,3}\,g_{4,1}-6\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{3,1}\,g_{4,1}+\frac{\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{3,1}\,g_{4,1}}{T^2}+\frac{2\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{3,1}\,g_{4,1}}{T}+$$

$$2\,T\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{3,1}\,g_{4,1}+T^2\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{3,1}\,g_{4,1}-2\,\epsilon^2\,g_{1,4}\,g_{2,3}\,g_{3,1}\,g_{4,1}+\frac{3\,\epsilon^2\,g_{1,4}\,g_{2,3}\,g_{3,1}\,g_{4,1}}{T}-$$

$$T\,\epsilon^2\,g_{1,4}\,g_{2,3}\,g_{3,1}\,g_{4,1}+6\,\epsilon^2\,g_{2,3}^2\,g_{3,1}\,g_{4,1}-\frac{\epsilon^2\,g_{2,3}^2\,g_{3,1}\,g_{4,1}}{T^2}-\frac{2\,\epsilon^2\,g_{2,3}^2\,g_{3,1}\,g_{4,1}}{T}-2\,T\,\epsilon^2\,g_{2,3}^2\,g_{3,1}\,g_{4,1}-$$

$$T^2\,\epsilon^2\,g_{2,3}^2\,g_{3,1}\,g_{4,1}-2\,\epsilon^2\,g_{1,3}\,g_{2,4}\,g_{3,1}\,g_{4,1}+\frac{3\,\epsilon^2\,g_{1,3}\,g_{2,4}\,g_{3,1}\,g_{4,1}}{T}-T\,\epsilon^2\,g_{1,3}\,g_{2,4}\,g_{3,1}\,g_{4,1}+$$

$$4\,\epsilon^2\,g_{2,3}\,g_{2,4}\,g_{3,1}\,g_{4,1}-\frac{6\,\epsilon^2\,g_{2,3}\,g_{2,4}\,g_{3,1}\,g_{4,1}}{T}+2\,T\,\epsilon^2\,g_{2,3}\,g_{2,4}\,g_{3,1}\,g_{4,1}-2\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{3,2}\,g_{4,1}-$$

$$\frac{\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{3,2}\,g_{4,1}}{T}+3\,T\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{3,2}\,g_{4,1}-2\,\epsilon^2\,g_{1,4}\,g_{2,3}\,g_{3,2}\,g_{4,1}+2\,\epsilon^2\,g_{2,3}^2\,g_{3,2}\,g_{4,1}+$$

$$\frac{\epsilon^2\,g_{2,3}^2\,g_{3,2}\,g_{4,1}}{T}-3\,T\,\epsilon^2\,g_{2,3}^2\,g_{3,2}\,g_{4,1}-2\,\epsilon^2\,g_{1,3}\,g_{2,4}\,g_{3,2}\,g_{4,1}+4\,\epsilon^2\,g_{2,3}\,g_{2,4}\,g_{3,2}\,g_{4,1}+$$

$$2\,\epsilon^2\,g_{1,3}\,g_{3,3}\,g_{4,1}+\frac{\epsilon^2\,g_{1,3}\,g_{3,3}\,g_{4,1}}{T}-3\,T\,\epsilon^2\,g_{1,3}\,g_{3,3}\,g_{4,1}+2\,\epsilon^2\,g_{1,4}\,g_{3,3}\,g_{4,1}-6\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,3}\,g_{4,1}+$$

$$\frac{\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,3}\,g_{4,1}}{T^2}+\frac{2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,3}\,g_{4,1}}{T}+2\,T\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,3}\,g_{4,1}+T^2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,3}\,g_{4,1}-$$

$$2\,\epsilon^2\,g_{1,4}\,g_{2,1}\,g_{3,3}\,g_{4,1}+\frac{3\,\epsilon^2\,g_{1,4}\,g_{2,1}\,g_{3,3}\,g_{4,1}}{T}-T\,\epsilon^2\,g_{1,4}\,g_{2,1}\,g_{3,3}\,g_{4,1}-2\,\epsilon^2\,g_{1,3}\,g_{2,2}\,g_{3,3}\,g_{4,1}-$$

$$\frac{\epsilon^2\,g_{1,3}\,g_{2,2}\,g_{3,3}\,g_{4,1}}{T}+3\,T\,\epsilon^2\,g_{1,3}\,g_{2,2}\,g_{3,3}\,g_{4,1}-2\,\epsilon^2\,g_{1,4}\,g_{2,2}\,g_{3,3}\,g_{4,1}-2\,\epsilon^2\,g_{2,3}\,g_{3,3}\,g_{4,1}-$$

$$\frac{\epsilon^2\,g_{2,3}\,g_{3,3}\,g_{4,1}}{T}+3\,T\,\epsilon^2\,g_{2,3}\,g_{3,3}\,g_{4,1}-6\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,3}\,g_{4,1}+\frac{\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,3}\,g_{4,1}}{T^2}+$$

$$\frac{2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,3}\,g_{4,1}}{T}+2\,T\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,3}\,g_{4,1}+T^2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,3}\,g_{4,1}-2\,\epsilon^2\,g_{1,2}\,g_{2,3}\,g_{3,3}\,g_{4,1}-$$

$$\frac{\epsilon^2\,g_{1,2}\,g_{2,3}\,g_{3,3}\,g_{4,1}}{T}+3\,T\,\epsilon^2\,g_{1,2}\,g_{2,3}\,g_{3,3}\,g_{4,1}+12\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,3}\,g_{4,1}-\frac{2\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,3}\,g_{4,1}}{T^2}-$$

$$\frac{4\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,3}\,g_{4,1}}{T}-4\,T\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,3}\,g_{4,1}-2\,T^2\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,3}\,g_{4,1}+4\,\epsilon^2\,g_{2,2}\,g_{2,3}\,g_{3,3}\,g_{4,1}+$$

$$\frac{2\,\epsilon^2\,g_{2,2}\,g_{2,3}\,g_{3,3}\,g_{4,1}}{T}-6\,T\,\epsilon^2\,g_{2,2}\,g_{2,3}\,g_{3,3}\,g_{4,1}-2\,\epsilon^2\,g_{2,4}\,g_{3,3}\,g_{4,1}-2\,\epsilon^2\,g_{1,1}\,g_{2,4}\,g_{3,3}\,g_{4,1}+$$

$$\frac{3\,\epsilon^2\,g_{1,1}\,g_{2,4}\,g_{3,3}\,g_{4,1}}{T} - T\,\epsilon^2\,g_{1,1}\,g_{2,4}\,g_{3,3}\,g_{4,1} - 2\,\epsilon^2\,g_{1,2}\,g_{2,4}\,g_{3,3}\,g_{4,1} + 4\,\epsilon^2\,g_{2,1}\,g_{2,4}\,g_{3,3}\,g_{4,1} -$$

$$\frac{6\,\epsilon^2\,g_{2,1}\,g_{2,4}\,g_{3,3}\,g_{4,1}}{T} + 2\,T\,\epsilon^2\,g_{2,1}\,g_{2,4}\,g_{3,3}\,g_{4,1} + 4\,\epsilon^2\,g_{2,2}\,g_{2,4}\,g_{3,3}\,g_{4,1} + 2\,\epsilon^2\,g_{1,3}\,g_{3,4}\,g_{4,1} -$$

$$2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,4}\,g_{4,1} + \frac{3\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,4}\,g_{4,1}}{T} - T\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,4}\,g_{4,1} - 2\,\epsilon^2\,g_{1,3}\,g_{2,2}\,g_{3,4}\,g_{4,1} -$$

$$2\,\epsilon^2\,g_{2,3}\,g_{3,4}\,g_{4,1} - 2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,4}\,g_{4,1} + \frac{3\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,4}\,g_{4,1}}{T} - T\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,4}\,g_{4,1} -$$

$$2\,\epsilon^2\,g_{1,2}\,g_{2,3}\,g_{3,4}\,g_{4,1} + 4\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,4}\,g_{4,1} - \frac{6\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,4}\,g_{4,1}}{T} + 2\,T\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,4}\,g_{4,1} +$$

$$4\,\epsilon^2\,g_{2,2}\,g_{2,3}\,g_{3,4}\,g_{4,1} + 6\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{4,1}^2 - \frac{\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{4,1}^2}{T^2} - \frac{2\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{4,1}^2}{T} - 2\,T\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{4,1}^2 -$$

$$T^2\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{4,1}^2 + 2\,\epsilon^2\,g_{1,4}\,g_{2,3}\,g_{4,1}^2 - \frac{3\,\epsilon^2\,g_{1,4}\,g_{2,3}\,g_{4,1}^2}{T} + T\,\epsilon^2\,g_{1,4}\,g_{2,3}\,g_{4,1}^2 - 6\,\epsilon^2\,g_{2,3}^2\,g_{4,1}^2 +$$

$$\frac{\epsilon^2\,g_{2,3}^2\,g_{4,1}^2}{T^2} + \frac{2\,\epsilon^2\,g_{2,3}^2\,g_{4,1}^2}{T} + 2\,T\,\epsilon^2\,g_{2,3}^2\,g_{4,1}^2 + T^2\,\epsilon^2\,g_{2,3}^2\,g_{4,1}^2 + 2\,\epsilon^2\,g_{1,3}\,g_{2,4}\,g_{4,1}^2 - \frac{3\,\epsilon^2\,g_{1,3}\,g_{2,4}\,g_{4,1}^2}{T} +$$

$$T\,\epsilon^2\,g_{1,3}\,g_{2,4}\,g_{4,1}^2 - 4\,\epsilon^2\,g_{2,3}\,g_{2,4}\,g_{4,1}^2 + \frac{6\,\epsilon^2\,g_{2,3}\,g_{2,4}\,g_{4,1}^2}{T} - 2\,T\,\epsilon^2\,g_{2,3}\,g_{2,4}\,g_{4,1}^2 - 2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{4,2} -$$

$$2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{4,2} + 4\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{4,2} - 2\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{3,1}\,g_{4,2} - \frac{\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{3,1}\,g_{4,2}}{T} +$$

$$3\,T\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{3,1}\,g_{4,2} - 2\,\epsilon^2\,g_{1,4}\,g_{2,3}\,g_{3,1}\,g_{4,2} + 2\,\epsilon^2\,g_{2,3}^2\,g_{3,1}\,g_{4,2} + \frac{\epsilon^2\,g_{2,3}^2\,g_{3,1}\,g_{4,2}}{T} -$$

$$3\,T\,\epsilon^2\,g_{2,3}^2\,g_{3,1}\,g_{4,2} - 2\,\epsilon^2\,g_{1,3}\,g_{2,4}\,g_{3,1}\,g_{4,2} + 4\,\epsilon^2\,g_{2,3}\,g_{2,4}\,g_{3,1}\,g_{4,2} - 2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,3}\,g_{4,2} -$$

$$\frac{\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,3}\,g_{4,2}}{T} + 3\,T\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,3}\,g_{4,2} - 2\,\epsilon^2\,g_{1,4}\,g_{2,1}\,g_{3,3}\,g_{4,2} - 2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,3}\,g_{4,2} -$$

$$\frac{\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,3}\,g_{4,2}}{T} + 3\,T\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,3}\,g_{4,2} + 4\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,3}\,g_{4,2} + \frac{2\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,3}\,g_{4,2}}{T} -$$

$$6\,T\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,3}\,g_{4,2} - 2\,\epsilon^2\,g_{1,1}\,g_{2,4}\,g_{3,3}\,g_{4,2} + 4\,\epsilon^2\,g_{2,1}\,g_{2,4}\,g_{3,3}\,g_{4,2} - 2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,4}\,g_{4,2} -$$

$$2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,4}\,g_{4,2} + 4\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,4}\,g_{4,2} + 4\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{4,1}\,g_{4,2} + \frac{2\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{4,1}\,g_{4,2}}{T} -$$

$$6\,T\,\epsilon^2\,g_{1,3}\,g_{2,3}\,g_{4,1}\,g_{4,2} + 4\,\epsilon^2\,g_{1,4}\,g_{2,3}\,g_{4,1}\,g_{4,2} - 4\,\epsilon^2\,g_{2,3}^2\,g_{4,1}\,g_{4,2} - \frac{2\,\epsilon^2\,g_{2,3}^2\,g_{4,1}\,g_{4,2}}{T} +$$

$$6\,T\,\epsilon^2\,g_{2,3}^2\,g_{4,1}\,g_{4,2} + 4\,\epsilon^2\,g_{1,3}\,g_{2,4}\,g_{4,1}\,g_{4,2} - 8\,\epsilon^2\,g_{2,3}\,g_{2,4}\,g_{4,1}\,g_{4,2} - \epsilon^2\,g_{4,3} + 2\,\epsilon^2\,g_{1,1}\,g_{4,3} -$$

$$2\,\epsilon^2\,g_{2,1}\,g_{4,3} - 2\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{4,3} + \frac{3\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{4,3}}{T} - T\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{4,3} - 2\,\epsilon^2\,g_{1,2}\,g_{2,1}\,g_{4,3} +$$

$$2\,\epsilon^2\,g_{2,1}^2\,g_{4,3} - \frac{3\,\epsilon^2\,g_{2,1}^2\,g_{4,3}}{T} + T\,\epsilon^2\,g_{2,1}^2\,g_{4,3} - 2\,\epsilon^2\,g_{1,1}\,g_{2,2}\,g_{4,3} + 4\,\epsilon^2\,g_{2,1}\,g_{2,2}\,g_{4,3} + 2\,\epsilon^2\,g_{1,3}\,g_{3,1}\,g_{4,3} +$$

$$\frac{\epsilon^2\,g_{1,3}\,g_{3,1}\,g_{4,3}}{T} - 3\,T\,\epsilon^2\,g_{1,3}\,g_{3,1}\,g_{4,3} + 2\,\epsilon^2\,g_{1,4}\,g_{3,1}\,g_{4,3} - 6\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,1}\,g_{4,3} + \frac{\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,1}\,g_{4,3}}{T^2} +$$

$$\frac{2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,1}\,g_{4,3}}{T}+2\,T\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,1}\,g_{4,3}+T^2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,1}\,g_{4,3}-2\,\epsilon^2\,g_{1,4}\,g_{2,1}\,g_{3,1}\,g_{4,3}+$$

$$\frac{3\,\epsilon^2\,g_{1,4}\,g_{2,1}\,g_{3,1}\,g_{4,3}}{T}-T\,\epsilon^2\,g_{1,4}\,g_{2,1}\,g_{3,1}\,g_{4,3}-2\,\epsilon^2\,g_{1,3}\,g_{2,2}\,g_{3,1}\,g_{4,3}-\frac{\epsilon^2\,g_{1,3}\,g_{2,2}\,g_{3,1}\,g_{4,3}}{T}+$$

$$3\,T\,\epsilon^2\,g_{1,3}\,g_{2,2}\,g_{3,1}\,g_{4,3}-2\,\epsilon^2\,g_{1,4}\,g_{2,2}\,g_{3,1}\,g_{4,3}-2\,\epsilon^2\,g_{2,3}\,g_{3,1}\,g_{4,3}-\frac{\epsilon^2\,g_{2,3}\,g_{3,1}\,g_{4,3}}{T}+$$

$$3\,T\,\epsilon^2\,g_{2,3}\,g_{3,1}\,g_{4,3}-6\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,1}\,g_{4,3}+\frac{\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,1}\,g_{4,3}}{T^2}+\frac{2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,1}\,g_{4,3}}{T}+$$

$$2\,T\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,1}\,g_{4,3}+T^2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,1}\,g_{4,3}-2\,\epsilon^2\,g_{1,2}\,g_{2,3}\,g_{3,1}\,g_{4,3}-\frac{\epsilon^2\,g_{1,2}\,g_{2,3}\,g_{3,1}\,g_{4,3}}{T}+$$

$$3\,T\,\epsilon^2\,g_{1,2}\,g_{2,3}\,g_{3,1}\,g_{4,3}+12\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,1}\,g_{4,3}-\frac{2\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,1}\,g_{4,3}}{T^2}-\frac{4\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,1}\,g_{4,3}}{T}-$$

$$4\,T\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,1}\,g_{4,3}-2\,T^2\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,1}\,g_{4,3}+4\,\epsilon^2\,g_{2,2}\,g_{2,3}\,g_{3,1}\,g_{4,3}+\frac{2\,\epsilon^2\,g_{2,2}\,g_{2,3}\,g_{3,1}\,g_{4,3}}{T}-$$

$$6\,T\,\epsilon^2\,g_{2,2}\,g_{2,3}\,g_{3,1}\,g_{4,3}-2\,\epsilon^2\,g_{2,4}\,g_{3,1}\,g_{4,3}-2\,\epsilon^2\,g_{1,1}\,g_{2,4}\,g_{3,1}\,g_{4,3}+\frac{3\,\epsilon^2\,g_{1,1}\,g_{2,4}\,g_{3,1}\,g_{4,3}}{T}-$$

$$T\,\epsilon^2\,g_{1,1}\,g_{2,4}\,g_{3,1}\,g_{4,3}-2\,\epsilon^2\,g_{1,2}\,g_{2,4}\,g_{3,1}\,g_{4,3}+4\,\epsilon^2\,g_{2,1}\,g_{2,4}\,g_{3,1}\,g_{4,3}-\frac{6\,\epsilon^2\,g_{2,1}\,g_{2,4}\,g_{3,1}\,g_{4,3}}{T}+$$

$$2\,T\,\epsilon^2\,g_{2,1}\,g_{2,4}\,g_{3,1}\,g_{4,3}+4\,\epsilon^2\,g_{2,2}\,g_{2,4}\,g_{3,1}\,g_{4,3}-2\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,2}\,g_{4,3}-\frac{\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,2}\,g_{4,3}}{T}+$$

$$3\,T\,\epsilon^2\,g_{1,3}\,g_{2,1}\,g_{3,2}\,g_{4,3}-2\,\epsilon^2\,g_{1,4}\,g_{2,1}\,g_{3,2}\,g_{4,3}-2\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,2}\,g_{4,3}-\frac{\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,2}\,g_{4,3}}{T}+$$

$$3\,T\,\epsilon^2\,g_{1,1}\,g_{2,3}\,g_{3,2}\,g_{4,3}+4\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,2}\,g_{4,3}+\frac{2\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,2}\,g_{4,3}}{T}-6\,T\,\epsilon^2\,g_{2,1}\,g_{2,3}\,g_{3,2}\,g_{4,3}-$$

$$2\,\epsilon^2\,g_{1,1}\,g_{2,4}\,g_{3,2}\,g_{4,3}+4\,\epsilon^2\,g_{2,1}\,g_{2,4}\,g_{3,2}\,g_{4,3}-\epsilon^2\,g_{3,3}\,g_{4,3}-\frac{\epsilon^2\,g_{3,3}\,g_{4,3}}{2\,T}+\frac{3}{2}\,T\,\epsilon^2\,g_{3,3}\,g_{4,3}+$$

$$2\,\epsilon^2\,g_{1,1}\,g_{3,3}\,g_{4,3}+\frac{\epsilon^2\,g_{1,1}\,g_{3,3}\,g_{4,3}}{T}-3\,T\,\epsilon^2\,g_{1,1}\,g_{3,3}\,g_{4,3}-2\,\epsilon^2\,g_{2,1}\,g_{3,3}\,g_{4,3}-\frac{\epsilon^2\,g_{2,1}\,g_{3,3}\,g_{4,3}}{T}+$$

$$3\,T\,\epsilon^2\,g_{2,1}\,g_{3,3}\,g_{4,3}-6\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{3,3}\,g_{4,3}+\frac{\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{3,3}\,g_{4,3}}{T^2}+\frac{2\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{3,3}\,g_{4,3}}{T}+$$

$$2\,T\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{3,3}\,g_{4,3}+T^2\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{3,3}\,g_{4,3}-2\,\epsilon^2\,g_{1,2}\,g_{2,1}\,g_{3,3}\,g_{4,3}-\frac{\epsilon^2\,g_{1,2}\,g_{2,1}\,g_{3,3}\,g_{4,3}}{T}+$$

$$3\,T\,\epsilon^2\,g_{1,2}\,g_{2,1}\,g_{3,3}\,g_{4,3}+6\,\epsilon^2\,g_{2,1}^2\,g_{3,3}\,g_{4,3}-\frac{\epsilon^2\,g_{2,1}^2\,g_{3,3}\,g_{4,3}}{T^2}-\frac{2\,\epsilon^2\,g_{2,1}^2\,g_{3,3}\,g_{4,3}}{T}-2\,T\,\epsilon^2\,g_{2,1}^2\,g_{3,3}\,g_{4,3}-$$

$$T^2\,\epsilon^2\,g_{2,1}^2\,g_{3,3}\,g_{4,3}-2\,\epsilon^2\,g_{1,1}\,g_{2,2}\,g_{3,3}\,g_{4,3}-\frac{\epsilon^2\,g_{1,1}\,g_{2,2}\,g_{3,3}\,g_{4,3}}{T}+3\,T\,\epsilon^2\,g_{1,1}\,g_{2,2}\,g_{3,3}\,g_{4,3}+$$

$$4\,\epsilon^2\,g_{2,1}\,g_{2,2}\,g_{3,3}\,g_{4,3}+\frac{2\,\epsilon^2\,g_{2,1}\,g_{2,2}\,g_{3,3}\,g_{4,3}}{T}-6\,T\,\epsilon^2\,g_{2,1}\,g_{2,2}\,g_{3,3}\,g_{4,3}-\epsilon^2\,g_{3,4}\,g_{4,3}+$$

$$2\,\epsilon^2\,g_{1,1}\,g_{3,4}\,g_{4,3}-2\,\epsilon^2\,g_{2,1}\,g_{3,4}\,g_{4,3}-2\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{3,4}\,g_{4,3}+\frac{3\,\epsilon^2\,g_{1,1}\,g_{2,1}\,g_{3,4}\,g_{4,3}}{T}-$$

$$T\epsilon^2 g_{1,1} g_{2,1} g_{3,4} g_{4,3} - 2\epsilon^2 g_{1,2} g_{2,1} g_{3,4} g_{4,3} + 2\epsilon^2 g_{2,1}^2 g_{3,4} g_{4,3} - \frac{3\epsilon^2 g_{2,1}^2 g_{3,4} g_{4,3}}{T} +$$
$$T\epsilon^2 g_{2,1}^2 g_{3,4} g_{4,3} - 2\epsilon^2 g_{1,1} g_{2,2} g_{3,4} g_{4,3} + 4\epsilon^2 g_{2,1} g_{2,2} g_{3,4} g_{4,3} - 4\epsilon^2 g_{1,3} g_{4,1} g_{4,3} -$$
$$\frac{2\epsilon^2 g_{1,3} g_{4,1} g_{4,3}}{T} + 6T\epsilon^2 g_{1,3} g_{4,1} g_{4,3} - 4\epsilon^2 g_{1,4} g_{4,1} g_{4,3} + 12\epsilon^2 g_{1,3} g_{2,1} g_{4,1} g_{4,3} -$$
$$\frac{2\epsilon^2 g_{1,3} g_{2,1} g_{4,1} g_{4,3}}{T^2} - \frac{4\epsilon^2 g_{1,3} g_{2,1} g_{4,1} g_{4,3}}{T} - 4T\epsilon^2 g_{1,3} g_{2,1} g_{4,1} g_{4,3} - 2T^2\epsilon^2 g_{1,3} g_{2,1} g_{4,1} g_{4,3} +$$
$$4\epsilon^2 g_{1,4} g_{2,1} g_{4,1} g_{4,3} - \frac{6\epsilon^2 g_{1,4} g_{2,1} g_{4,1} g_{4,3}}{T} + 2T\epsilon^2 g_{1,4} g_{2,1} g_{4,1} g_{4,3} + 4\epsilon^2 g_{1,3} g_{2,2} g_{4,1} g_{4,3} +$$
$$\frac{2\epsilon^2 g_{1,3} g_{2,2} g_{4,1} g_{4,3}}{T} - 6T\epsilon^2 g_{1,3} g_{2,2} g_{4,1} g_{4,3} + 4\epsilon^2 g_{1,4} g_{2,2} g_{4,1} g_{4,3} + 4\epsilon^2 g_{2,3} g_{4,1} g_{4,3} +$$
$$\frac{2\epsilon^2 g_{2,3} g_{4,1} g_{4,3}}{T} - 6T\epsilon^2 g_{2,3} g_{4,1} g_{4,3} + 12\epsilon^2 g_{1,1} g_{2,3} g_{4,1} g_{4,3} - \frac{2\epsilon^2 g_{1,1} g_{2,3} g_{4,1} g_{4,3}}{T^2} -$$
$$\frac{4\epsilon^2 g_{1,1} g_{2,3} g_{4,1} g_{4,3}}{T} - 4T\epsilon^2 g_{1,1} g_{2,3} g_{4,1} g_{4,3} - 2T^2\epsilon^2 g_{1,1} g_{2,3} g_{4,1} g_{4,3} + 4\epsilon^2 g_{1,2} g_{2,3} g_{4,1} g_{4,3} +$$
$$\frac{2\epsilon^2 g_{1,2} g_{2,3} g_{4,1} g_{4,3}}{T} - 6T\epsilon^2 g_{1,2} g_{2,3} g_{4,1} g_{4,3} - 24\epsilon^2 g_{2,1} g_{2,3} g_{4,1} g_{4,3} + \frac{4\epsilon^2 g_{2,1} g_{2,3} g_{4,1} g_{4,3}}{T^2} +$$
$$\frac{8\epsilon^2 g_{2,1} g_{2,3} g_{4,1} g_{4,3}}{T} + 8T\epsilon^2 g_{2,1} g_{2,3} g_{4,1} g_{4,3} + 4T^2\epsilon^2 g_{2,1} g_{2,3} g_{4,1} g_{4,3} - 8\epsilon^2 g_{2,2} g_{2,3} g_{4,1} g_{4,3} -$$
$$\frac{4\epsilon^2 g_{2,2} g_{2,3} g_{4,1} g_{4,3}}{T} + 12T\epsilon^2 g_{2,2} g_{2,3} g_{4,1} g_{4,3} + 4\epsilon^2 g_{2,4} g_{4,1} g_{4,3} + 4\epsilon^2 g_{1,1} g_{2,4} g_{4,1} g_{4,3} -$$
$$\frac{6\epsilon^2 g_{1,1} g_{2,4} g_{4,1} g_{4,3}}{T} + 2T\epsilon^2 g_{1,1} g_{2,4} g_{4,1} g_{4,3} + 4\epsilon^2 g_{1,2} g_{2,4} g_{4,1} g_{4,3} - 8\epsilon^2 g_{2,1} g_{2,4} g_{4,1} g_{4,3} +$$
$$\frac{12\epsilon^2 g_{2,1} g_{2,4} g_{4,1} g_{4,3}}{T} - 4T\epsilon^2 g_{2,1} g_{2,4} g_{4,1} g_{4,3} - 8\epsilon^2 g_{2,2} g_{2,4} g_{4,1} g_{4,3} + 4\epsilon^2 g_{1,3} g_{2,1} g_{4,2} g_{4,3} +$$
$$\frac{2\epsilon^2 g_{1,3} g_{2,1} g_{4,2} g_{4,3}}{T} - 6T\epsilon^2 g_{1,3} g_{2,1} g_{4,2} g_{4,3} + 4\epsilon^2 g_{1,4} g_{2,1} g_{4,2} g_{4,3} + 4\epsilon^2 g_{1,1} g_{2,3} g_{4,2} g_{4,3} +$$
$$\frac{2\epsilon^2 g_{1,1} g_{2,3} g_{4,2} g_{4,3}}{T} - 6T\epsilon^2 g_{1,1} g_{2,3} g_{4,2} g_{4,3} - 8\epsilon^2 g_{2,1} g_{2,3} g_{4,2} g_{4,3} - \frac{4\epsilon^2 g_{2,1} g_{2,3} g_{4,2} g_{4,3}}{T} +$$
$$12T\epsilon^2 g_{2,1} g_{2,3} g_{4,2} g_{4,3} + 4\epsilon^2 g_{1,1} g_{2,4} g_{4,2} g_{4,3} - 8\epsilon^2 g_{2,1} g_{2,4} g_{4,2} g_{4,3} + \epsilon^2 g_{4,3}^2 + \frac{\epsilon^2 g_{4,3}^2}{2T} -$$
$$\frac{3}{2} T\epsilon^2 g_{4,3}^2 - 2\epsilon^2 g_{1,1} g_{4,3}^2 - \frac{\epsilon^2 g_{1,1} g_{4,3}^2}{T} + 3T\epsilon^2 g_{1,1} g_{4,3}^2 + 2\epsilon^2 g_{2,1} g_{4,3}^2 + \frac{\epsilon^2 g_{2,1} g_{4,3}^2}{T} -$$
$$3T\epsilon^2 g_{2,1} g_{4,3}^2 + 6\epsilon^2 g_{1,1} g_{2,1} g_{4,3}^2 - \frac{\epsilon^2 g_{1,1} g_{2,1} g_{4,3}^2}{T^2} - \frac{2\epsilon^2 g_{1,1} g_{2,1} g_{4,3}^2}{T} - 2T\epsilon^2 g_{1,1} g_{2,1} g_{4,3}^2 -$$
$$T^2\epsilon^2 g_{1,1} g_{2,1} g_{4,3}^2 + 2\epsilon^2 g_{1,2} g_{2,1} g_{4,3}^2 + \frac{\epsilon^2 g_{1,2} g_{2,1} g_{4,3}^2}{T} - 3T\epsilon^2 g_{1,2} g_{2,1} g_{4,3}^2 - 6\epsilon^2 g_{2,1}^2 g_{4,3}^2 +$$
$$\frac{\epsilon^2 g_{2,1}^2 g_{4,3}^2}{T^2} + \frac{2\epsilon^2 g_{2,1}^2 g_{4,3}^2}{T} + 2T\epsilon^2 g_{2,1}^2 g_{4,3}^2 + T^2\epsilon^2 g_{2,1}^2 g_{4,3}^2 + 2\epsilon^2 g_{1,1} g_{2,2} g_{4,3}^2 +$$

$$\frac{\epsilon^2 g_{1,1} g_{2,2} g_{4,3}^2}{T} - 3 T \epsilon^2 g_{1,1} g_{2,2} g_{4,3}^2 - 4 \epsilon^2 g_{2,1} g_{2,2} g_{4,3}^2 - \frac{2 \epsilon^2 g_{2,1} g_{2,2} g_{4,3}^2}{T} + 6 T \epsilon^2 g_{2,1} g_{2,2} g_{4,3}^2 +$$
$$2 \epsilon^2 g_{1,3} g_{3,1} g_{4,4} - 2 \epsilon^2 g_{1,3} g_{2,1} g_{3,1} g_{4,4} + \frac{3 \epsilon^2 g_{1,3} g_{2,1} g_{3,1} g_{4,4}}{T} - T \epsilon^2 g_{1,3} g_{2,1} g_{3,1} g_{4,4} -$$
$$2 \epsilon^2 g_{1,3} g_{2,2} g_{3,1} g_{4,4} - 2 \epsilon^2 g_{2,3} g_{3,1} g_{4,4} - 2 \epsilon^2 g_{1,1} g_{2,3} g_{3,1} g_{4,4} + \frac{3 \epsilon^2 g_{1,1} g_{2,3} g_{3,1} g_{4,4}}{T} -$$
$$T \epsilon^2 g_{1,1} g_{2,3} g_{3,1} g_{4,4} - 2 \epsilon^2 g_{1,2} g_{2,3} g_{3,1} g_{4,4} + 4 \epsilon^2 g_{2,1} g_{2,3} g_{3,1} g_{4,4} - \frac{6 \epsilon^2 g_{2,1} g_{2,3} g_{3,1} g_{4,4}}{T} +$$
$$2 T \epsilon^2 g_{2,1} g_{2,3} g_{3,1} g_{4,4} + 4 \epsilon^2 g_{2,2} g_{2,3} g_{3,1} g_{4,4} - 2 \epsilon^2 g_{1,3} g_{2,1} g_{3,2} g_{4,4} - 2 \epsilon^2 g_{1,1} g_{2,3} g_{3,2} g_{4,4} +$$
$$4 \epsilon^2 g_{2,1} g_{2,3} g_{3,2} g_{4,4} - \epsilon^2 g_{3,3} g_{4,4} + 2 \epsilon^2 g_{1,1} g_{3,3} g_{4,4} - 2 \epsilon^2 g_{2,1} g_{3,3} g_{4,4} - 2 \epsilon^2 g_{1,1} g_{2,1} g_{3,3} g_{4,4} +$$
$$\frac{3 \epsilon^2 g_{1,1} g_{2,1} g_{3,3} g_{4,4}}{T} - T \epsilon^2 g_{1,1} g_{2,1} g_{3,3} g_{4,4} - 2 \epsilon^2 g_{1,2} g_{2,1} g_{3,3} g_{4,4} + 2 \epsilon^2 g_{2,1}^2 g_{3,3} g_{4,4} -$$
$$\frac{3 \epsilon^2 g_{2,1}^2 g_{3,3} g_{4,4}}{T} + T \epsilon^2 g_{2,1}^2 g_{3,3} g_{4,4} - 2 \epsilon^2 g_{1,1} g_{2,2} g_{3,3} g_{4,4} + 4 \epsilon^2 g_{2,1} g_{2,2} g_{3,3} g_{4,4} -$$
$$4 \epsilon^2 g_{1,3} g_{4,1} g_{4,4} + 4 \epsilon^2 g_{1,3} g_{2,1} g_{4,1} g_{4,4} - \frac{6 \epsilon^2 g_{1,3} g_{2,1} g_{4,1} g_{4,4}}{T} + 2 T \epsilon^2 g_{1,3} g_{2,1} g_{4,1} g_{4,4} +$$
$$4 \epsilon^2 g_{1,3} g_{2,2} g_{4,1} g_{4,4} + 4 \epsilon^2 g_{2,3} g_{4,1} g_{4,4} + 4 \epsilon^2 g_{1,1} g_{2,3} g_{4,1} g_{4,4} - \frac{6 \epsilon^2 g_{1,1} g_{2,3} g_{4,1} g_{4,4}}{T} +$$
$$2 T \epsilon^2 g_{1,1} g_{2,3} g_{4,1} g_{4,4} + 4 \epsilon^2 g_{1,2} g_{2,3} g_{4,1} g_{4,4} - 8 \epsilon^2 g_{2,1} g_{2,3} g_{4,1} g_{4,4} + \frac{12 \epsilon^2 g_{2,1} g_{2,3} g_{4,1} g_{4,4}}{T} -$$
$$4 T \epsilon^2 g_{2,1} g_{2,3} g_{4,1} g_{4,4} - 8 \epsilon^2 g_{2,2} g_{2,3} g_{4,1} g_{4,4} + 4 \epsilon^2 g_{1,3} g_{2,1} g_{4,2} g_{4,4} + 4 \epsilon^2 g_{1,1} g_{2,3} g_{4,2} g_{4,4} -$$
$$8 \epsilon^2 g_{2,1} g_{2,3} g_{4,2} g_{4,4} + 2 \epsilon^2 g_{4,3} g_{4,4} - 4 \epsilon^2 g_{1,1} g_{4,3} g_{4,4} + 4 \epsilon^2 g_{2,1} g_{4,3} g_{4,4} + 4 \epsilon^2 g_{1,1} g_{2,1} g_{4,3} g_{4,4} -$$
$$\frac{6 \epsilon^2 g_{1,1} g_{2,1} g_{4,3} g_{4,4}}{T} + 2 T \epsilon^2 g_{1,1} g_{2,1} g_{4,3} g_{4,4} + 4 \epsilon^2 g_{1,2} g_{2,1} g_{4,3} g_{4,4} - 4 \epsilon^2 g_{2,1}^2 g_{4,3} g_{4,4} +$$
$$\frac{6 \epsilon^2 g_{2,1}^2 g_{4,3} g_{4,4}}{T} - 2 T \epsilon^2 g_{2,1}^2 g_{4,3} g_{4,4} + 4 \epsilon^2 g_{1,1} g_{2,2} g_{4,3} g_{4,4} - 8 \epsilon^2 g_{2,1} g_{2,2} g_{4,3} g_{4,4}$$

(Alt) In[ ]:=

```
Z3 = Expand[Z2 /. F[fs_, es_] :> Expand[gPair[
       Replace[fs, Thread[es -> Range@Length@es], {2}], Length@es
     ] /. g_α_,β_ :> G[[es[[α]], es[[β]]]] ] ]
```

(Alt) Out[ ]=

$$1 - \frac{T^2 \epsilon}{(1-3T+T^2)^2} + \frac{3 T^3 \epsilon}{(1-3T+T^2)^2} - \frac{T^4 \epsilon}{(1-3T+T^2)^2} + \frac{T^2 \epsilon}{1-3T+T^2} + \frac{5 T^2 \epsilon^2}{(1-3T+T^2)^4} - \frac{36 T^3 \epsilon^2}{(1-3T+T^2)^4} +$$
$$\frac{103 T^4 \epsilon^2}{(1-3T+T^2)^4} - \frac{156 T^5 \epsilon^2}{(1-3T+T^2)^4} + \frac{134 T^6 \epsilon^2}{(1-3T+T^2)^4} - \frac{57 T^7 \epsilon^2}{(1-3T+T^2)^4} + \frac{9 T^8 \epsilon^2}{(1-3T+T^2)^4} + \frac{2 T \epsilon^2}{(1-3T+T^2)^3} -$$
$$\frac{19 T^2 \epsilon^2}{(1-3T+T^2)^3} + \frac{61 T^3 \epsilon^2}{(1-3T+T^2)^3} - \frac{86 T^4 \epsilon^2}{(1-3T+T^2)^3} + \frac{63 T^5 \epsilon^2}{(1-3T+T^2)^3} - \frac{16 T^6 \epsilon^2}{(1-3T+T^2)^3} - \frac{3 T \epsilon^2}{(1-3T+T^2)^2} +$$
$$\frac{25 T^2 \epsilon^2}{2(1-3T+T^2)^2} - \frac{29 T^3 \epsilon^2}{2(1-3T+T^2)^2} + \frac{15 T^4 \epsilon^2}{2(1-3T+T^2)^2} + \frac{T \epsilon^2}{1-3T+T^2} - \frac{T^2 \epsilon^2}{2(1-3T+T^2)}$$

In[•]:=
```
Z₃[Knot[3, 1]] // Timing
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=
$$\left\{49.9844,\ \left\{1+z^2,\ 1+\left(2z^2+z^4\right)\in+\left(2-4z^2+3z^4+4z^6+z^8\right)\in^2+\left(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12}\right)\in^3\right\}\right\}$$

In[•]:=
```
Z₃[Knot[3, 1]] // Timing
```

Out[•]=
$$\left\{1.26563,\ \left\{1+z^2,\ 1+\left(2z^2+z^4\right)\in+\left(2-4z^2+3z^4+4z^6+z^8\right)\in^2+\left(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12}\right)\in^3\right\}\right\}$$

Demos

exec
```
nb2tex$PDFWidth = 8 / 0.75;
```

tex
```
\end{multicols}
\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.75]{#1}}
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.75]{#1}}
```

```
In[•]:= GST48 = EPD[X14,1, X̄2,29, X3,40, X43,4, X̄26,5, X6,95, X96,7, X13,8, X̄9,28, X10,41, X42,11, X̄27,12,
          X30,15, X̄16,61, X̄17,72, X̄18,83, X19,34, X̄89,20, X̄21,92, X̄79,22, X̄68,23, X̄57,24, X̄25,56, X62,31,
          X73,32, X84,33, X̄50,35, X36,81, X37,70, X38,59, X̄39,54, X44,55, X58,45, X69,46, X80,47, X48,91,
          X90,49, X51,82, X52,71, X53,60, X̄63,74, X̄64,85, X̄76,65, X̄87,66, X̄67,94, X̄75,86, X̄88,77, X̄78,93];
        Z2[GST48] // Timing
        Z2[GST48] // Timing
```

Out[•]=

$$\{564.578,\ \{1-4z^2-61z^4-207z^6-296z^8-210z^{10}-77z^{12}-14z^{14}-z^{16},$$
$$1+\left(38z^2+255z^4+1696z^6+16\,281z^8+86\,952z^{10}+259\,994z^{12}+487\,372z^{14}+615\,066z^{16}+\right.$$
$$\left.543\,148z^{18}+341\,714z^{20}+153\,722z^{22}+48\,983z^{24}+10\,776z^{26}+1554z^{28}+132z^{30}+5z^{32}\right)\in+$$
$$\left(-8-484z^2+9709z^4+165\,952z^6+1\,590\,491z^8+16\,256\,508z^{10}+115\,341\,797z^{12}+432\,685\,748z^{14}+\right.$$
$$395\,838\,354z^{16}-4\,017\,557\,792z^{18}-23\,300\,064\,167z^{20}-70\,082\,264\,972z^{22}-142\,572\,271\,191z^{24}-$$
$$209\,475\,503\,700z^{26}-221\,616\,295\,209z^{28}-151\,502\,648\,428z^{30}-23\,700\,199\,243z^{32}+$$
$$99\,462\,146\,328z^{34}+164\,920\,463\,074z^{36}+162\,550\,825\,432z^{38}+119\,164\,552\,296z^{40}+$$
$$69\,153\,062\,608z^{42}+32\,547\,596\,611z^{44}+12\,541\,195\,448z^{46}+3\,961\,384\,155z^{48}+1\,021\,219\,696z^{50}+$$
$$\left.212\,773\,106z^{52}+35\,264\,208z^{54}+4\,537\,548z^{56}+436\,600z^{58}+29\,536z^{60}+1252z^{62}+25z^{64}\right)\in^2\}\}$$

Out[•]=

$$\{598.109,\ \{1-4z^2-61z^4-207z^6-296z^8-210z^{10}-77z^{12}-14z^{14}-z^{16},$$
$$1+\left(38z^2+255z^4+1696z^6+16\,281z^8+86\,952z^{10}+259\,994z^{12}+487\,372z^{14}+615\,066z^{16}+\right.$$
$$\left.543\,148z^{18}+341\,714z^{20}+153\,722z^{22}+48\,983z^{24}+10\,776z^{26}+1554z^{28}+132z^{30}+5z^{32}\right)\in+$$
$$\left(-8-484z^2+9709z^4+165\,952z^6+1\,590\,491z^8+16\,256\,508z^{10}+115\,341\,797z^{12}+432\,685\,748z^{14}+\right.$$
$$395\,838\,354z^{16}-4\,017\,557\,792z^{18}-23\,300\,064\,167z^{20}-70\,082\,264\,972z^{22}-142\,572\,271\,191z^{24}-$$
$$209\,475\,503\,700z^{26}-221\,616\,295\,209z^{28}-151\,502\,648\,428z^{30}-23\,700\,199\,243z^{32}+$$
$$99\,462\,146\,328z^{34}+164\,920\,463\,074z^{36}+162\,550\,825\,432z^{38}+119\,164\,552\,296z^{40}+$$
$$69\,153\,062\,608z^{42}+32\,547\,596\,611z^{44}+12\,541\,195\,448z^{46}+3\,961\,384\,155z^{48}+1\,021\,219\,696z^{50}+$$
$$\left.212\,773\,106z^{52}+35\,264\,208z^{54}+4\,537\,548z^{56}+436\,600z^{58}+29\,536z^{60}+1252z^{62}+25z^{64}\right)\in^2\}\}$$

pdf

```
        Z2[GST48] (* takes a few minutes *)
```

Out[•]=
pdf

$$\{1-4z^2-61z^4-207z^6-296z^8-210z^{10}-77z^{12}-14z^{14}-z^{16},$$
$$1+\left(38z^2+255z^4+1696z^6+16\,281z^8+86\,952z^{10}+259\,994z^{12}+487\,372z^{14}+615\,066z^{16}+\right.$$
$$\left.543\,148z^{18}+341\,714z^{20}+153\,722z^{22}+48\,983z^{24}+10\,776z^{26}+1554z^{28}+132z^{30}+5z^{32}\right)\in+$$
$$\left(-8-484z^2+9709z^4+165\,952z^6+1\,590\,491z^8+16\,256\,508z^{10}+115\,341\,797z^{12}+432\,685\,748z^{14}+\right.$$
$$395\,838\,354z^{16}-4\,017\,557\,792z^{18}-23\,300\,064\,167z^{20}-70\,082\,264\,972z^{22}-142\,572\,271\,191z^{24}-$$
$$209\,475\,503\,700z^{26}-221\,616\,295\,209z^{28}-151\,502\,648\,428z^{30}-23\,700\,199\,243z^{32}+$$
$$99\,462\,146\,328z^{34}+164\,920\,463\,074z^{36}+162\,550\,825\,432z^{38}+119\,164\,552\,296z^{40}+$$
$$69\,153\,062\,608z^{42}+32\,547\,596\,611z^{44}+12\,541\,195\,448z^{46}+3\,961\,384\,155z^{48}+1\,021\,219\,696z^{50}+$$
$$\left.212\,773\,106z^{52}+35\,264\,208z^{54}+4\,537\,548z^{56}+436\,600z^{58}+29\,536z^{60}+1252z^{62}+25z^{64}\right)\in^2\}$$

```
In[•]:= Table[Join[{K⟦1⟧_K⟦2⟧}, Z_3[K]], {K, AllKnots[{3, 6}]}] // Timing
```

```
Out[•]=
{256.063, {{3_1, 1 + z^2,
    1 + (2 z^2 + z^4) ∈ + (2 - 4 z^2 + 3 z^4 + 4 z^6 + z^8) ∈^2 + (-12 + 74 z^2 - 27 z^4 - 20 z^6 + 8 z^8 + 6 z^10 + z^12) ∈^3},
  {4_1, 1 - z^2, 1 + (-2 + 2 z^4) ∈^2}, {5_1, 1 + 3 z^2 + z^4, 1 + (10 z^2 + 21 z^4 + 12 z^6 + 2 z^8) ∈ +
    (6 - 28 z^2 + 33 z^4 + 364 z^6 + 655 z^8 + 536 z^10 + 227 z^12 + 48 z^14 + 4 z^16) ∈^2 +
    (-60 + 970 z^2 + 645 z^4 - 3380 z^6 - 3280 z^8 + 7470 z^10 + 19475 z^12 +
       20536 z^14 + 12564 z^16 + 4774 z^18 + 1109 z^20 + 144 z^22 + 8 z^24) ∈^3},
  {5_2, 1 + 2 z^2, 1 + (6 z^2 + 5 z^4) ∈ + (4 - 20 z^2 + 43 z^4 + 64 z^6 + 26 z^8) ∈^2 +
    (-36 + 498 z^2 - 883 z^4 + 100 z^6 + 816 z^8 + 556 z^10 + 146 z^12) ∈^3},
  {6_1, 1 - 2 z^2, 1 + (-2 z^2 + z^4) ∈ + (-4 + 4 z^2 + 25 z^4 - 8 z^6 + 2 z^8) ∈^2 +
    (12 + 154 z^2 - 223 z^4 - 608 z^6 + 100 z^8 - 52 z^10 + 10 z^12) ∈^3},
  {6_2, 1 - z^2 - z^4, 1 + (-2 z^2 - 3 z^4 + 2 z^6 + z^8) ∈ +
    (-2 - 4 z^2 + 29 z^4 + 28 z^6 + 42 z^8 - 8 z^10 - 2 z^12 + 4 z^14 + z^16) ∈^2 + (12 + 166 z^2 + 155 z^4 -
       194 z^6 - 2453 z^8 - 1622 z^10 - 1967 z^12 - 258 z^14 + 49 z^16 - 30 z^18 + z^20 + 6 z^22 + z^24) ∈^3},
  {6_3, 1 + z^2 + z^4, 1 + (2 + 8 z^2 - 16 z^6 - 24 z^8 - 16 z^10 - 2 z^12) ∈^2}}}
```

```
In[•]:= Table[Join[{K⟦1⟧_K⟦2⟧}, Z_3[K]], {K, AllKnots[{3, 6}]}] // Timing
```

```
Out[•]=
{143.641, {{3_1, 1 + z^2,
    1 + (2 z^2 + z^4) ∈ + (2 - 4 z^2 + 3 z^4 + 4 z^6 + z^8) ∈^2 + (-12 + 74 z^2 - 27 z^4 - 20 z^6 + 8 z^8 + 6 z^10 + z^12) ∈^3},
  {4_1, 1 - z^2, 1 + (-2 + 2 z^4) ∈^2}, {5_1, 1 + 3 z^2 + z^4, 1 + (10 z^2 + 21 z^4 + 12 z^6 + 2 z^8) ∈ +
    (6 - 28 z^2 + 33 z^4 + 364 z^6 + 655 z^8 + 536 z^10 + 227 z^12 + 48 z^14 + 4 z^16) ∈^2 +
    (-60 + 970 z^2 + 645 z^4 - 3380 z^6 - 3280 z^8 + 7470 z^10 + 19475 z^12 +
       20536 z^14 + 12564 z^16 + 4774 z^18 + 1109 z^20 + 144 z^22 + 8 z^24) ∈^3},
  {5_2, 1 + 2 z^2, 1 + (6 z^2 + 5 z^4) ∈ + (4 - 20 z^2 + 43 z^4 + 64 z^6 + 26 z^8) ∈^2 +
    (-36 + 498 z^2 - 883 z^4 + 100 z^6 + 816 z^8 + 556 z^10 + 146 z^12) ∈^3},
  {6_1, 1 - 2 z^2, 1 + (-2 z^2 + z^4) ∈ + (-4 + 4 z^2 + 25 z^4 - 8 z^6 + 2 z^8) ∈^2 +
    (12 + 154 z^2 - 223 z^4 - 608 z^6 + 100 z^8 - 52 z^10 + 10 z^12) ∈^3},
  {6_2, 1 - z^2 - z^4, 1 + (-2 z^2 - 3 z^4 + 2 z^6 + z^8) ∈ +
    (-2 - 4 z^2 + 29 z^4 + 28 z^6 + 42 z^8 - 8 z^10 - 2 z^12 + 4 z^14 + z^16) ∈^2 + (12 + 166 z^2 + 155 z^4 -
       194 z^6 - 2453 z^8 - 1622 z^10 - 1967 z^12 - 258 z^14 + 49 z^16 - 30 z^18 + z^20 + 6 z^22 + z^24) ∈^3},
  {6_3, 1 + z^2 + z^4, 1 + (2 + 8 z^2 - 16 z^6 - 24 z^8 - 16 z^10 - 2 z^12) ∈^2}}}
```

```
tex
```

```
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[width=\linewidth]{#1}}
```

pdf

```
TableForm[Table[Join[{K⟦1⟧_K⟦2⟧}, Z_3[K]], {K, AllKnots[{3, 6}]}],
 TableAlignments → Center] (* takes a few minutes *)
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[#]//TableForm=
pdf

3_1	$1+z^2$	$1+\left(2z^2+z^4\right)$
4_1	$1-z^2$	
5_1	$1+3z^2+z^4$	$1+\left(10z^2+21z^4+12z^6+2z^8\right)\in+\left(6-28z^2+33z^4+364z^6+655z^8+536z^{10}+227$
5_2	$1+2z^2$	$1+\left(6z^2+5z^4\right)\in+\left(4-$
6_1	$1-2z^2$	$1+\left(-2z^2+z^4\right)\in+\left($
6_2	$1-z^2-z^4$	$1+\left(-2z^2-3z^4+2z^6+z^8\right)\in+\left(-2-4z^2+29z^4+28z^6+42z^8-$
6_3	$1+z^2+z^4$	