Pensieve header: Implementing $\Theta$ - the main notebook accompanying Talks/MonteVerita-2604.

## Invariance under R3

exec

```
nb2tex$TeXFileName = "Invariance.tex";
```

This is Theta.nb of http://drorbn.net/mv26/ap.

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\MonteVerita-2604"];
```

pdf

In[∘]:=
```
T_3 = T_1 T_2;
```

pdf

In[∘]:=
```
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```

pdf

In[∘]:=
```
F_1[{s_, i_, j_}] =
  CF[s (1/2 - g_3ii + T_2^s g_1ii g_2ji - g_1ii g_2jj - (T_2^s - 1) g_2ji g_3ii + 2 g_2jj g_3ii - (1 - T_3^s) g_2ji g_3ji -
      g_2ii g_3jj - T_2^s g_2ji g_3jj + g_1ii g_3jj + ((T_1^s - 1) g_1ji (T_2^(2 s) g_2ji - T_2^s g_2jj + T_2^s g_3jj) +
        (T_3^s - 1) g_3ji (1 - T_2^s g_1ii - (T_1^s - 1) (T_2^s + 1) g_1ji + (T_2^s - 2) g_2jj + g_2ij)) / (T_2^s - 1))];
```

pdf

In[∘]:=
```
F_2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] := CF[
  s1 (T_1^s0 - 1) (T_2^s1 - 1)^-1 (T_3^s1 - 1) g_{1,j1,i0} g_{3,j0,i1} ( (T_2^s0 g_{2,i1,i0} - g_{2,i1,j0}) - (T_2^s0 g_{2,j1,i0} - g_{2,j1,j0}))]
```

pdf

In[∘]:=
```
F_3[φ_, k_] = -φ/2 + φ g_3kk;
```

pdf

```
δ_{i_,j_} := If[i === j, 1, 0];
gR_{s_,i_,j_} := {
  g_{ν_ jβ_} :→ g_{ν j⁺ β} + δ_{jβ}, g_{ν_ iβ_} :→ T_ν^s g_{ν i⁺ β} + (1 - T_ν^s) g_{ν j⁺ β} + δ_{iβ},
  g_{ν_ α_ i⁺} :→ T_ν^s g_{ναi} + δ_{αi⁺}, g_{ν_ α_ j⁺} :→ g_{ναj} + (1 - T_ν^s) g_{ναi} + δ_{αj⁺}
 }
```

Proof of Reidemeister 3:

pdf

In[∘]:=
```
DSum[Cs___] := Sum[F_1[c], {c, {Cs}}] + Sum[F_2[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
lhs = DSum[{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}, {s, m, n}] //. gR_{1,j,k} ⋃ gR_{1,i,k⁺} ⋃ gR_{1,i⁺,j⁺};
rhs = DSum[{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}, {s, m, n}] //. gR_{1,i,j} ⋃ gR_{1,i⁺,k} ⋃ gR_{1,j⁺,k⁺};
Simplify[lhs == rhs]
```

Out[∘]=
pdf

```
True
```

exec

```
nb2tex$TeXFileName = "Program.tex";
nb2tex$PDFWidth = 4.2
```

## The Main Program

tex

{\red\bf The Main Program.}

pdf

```
In[∘]:= Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/v25/ap to compute rotation numbers.
```

pdf

```
Loading PolyPlot.m from http://drorbn.net/mv26/ap to plot 2-variable polynomials.
```

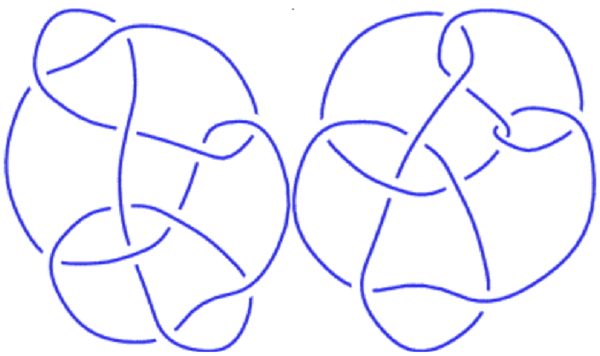
pdf

```
In[∘]:= Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, θ},
    {Cs, φ} = Rot[K]; n = Length[Cs];
    A = IdentityMatrix[2 n + 1];
    Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                                                  0     -1    ))];
    Δ = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
    G = Inverse[A];
    ev[ℰ_] := Factor[ℰ /. g_{ν_, α_, β_} :> (G[[α, β]] /. T → T_ν)];
    θ = ev[∑_{k=1}^{n} F_1[Cs[[k]]]];
    θ += ev[∑_{k1=1}^{n} ∑_{k2=1}^{n} F_2[Cs[[k1]], Cs[[k2]]]];
    θ += ev[∑_{k=1}^{2 n} F_3[φ[[k]], k]];
    Factor@{Δ, (Δ /. T → T_1) (Δ /. T → T_2) (Δ /. T → T_3) θ}];
```

## The Trefoil, Conway, and Kinoshita-Terasaka Knots

tex

\vskip 3mm
{\red\bf The Trefoil, Conway, and Kinoshita-Terasaka Knots.}
\parpic[r]{\parbox{15mm}{
 \includegraphics[width=15mm]{../PhuQuoc-2506/K11n34.png}
 \vskip 1mm
 \includegraphics[width=15mm]{../PhuQuoc-2506/K11n42.png}
}}



pdf

In[∘]:=
```
Θ[Knot[3, 1]] // Expand
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

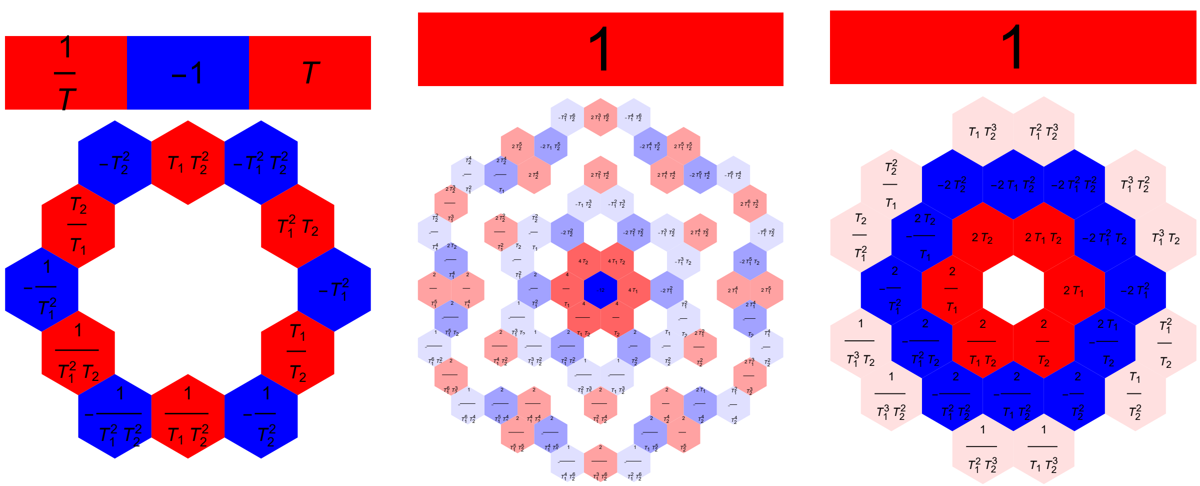
Out[∘]=
pdf

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2^2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}$$

pdf

In[∘]:=
```
GraphicsRow[PolyPlot[Θ[Knot[#]], Labeled → True] & /@ {"3_1", "K11n34", "K11n42"}]
```

Out[∘]=
pdf




tex

(Note that the genus of the Conway knot appears to be bigger than the genus of Kinoshita-Terasaka)

## Some Torus Knots

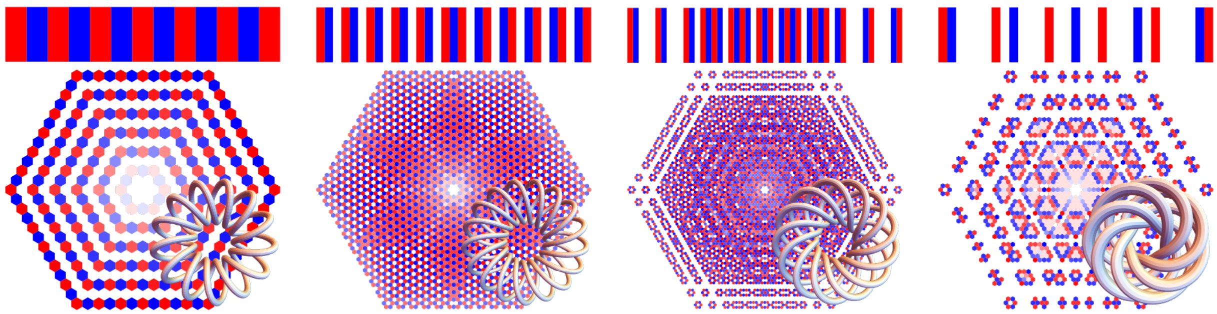
tex

{\red\bf Some Torus Knots.}

*pdf*

In[∘]:=
```
GraphicsRow[ImageCompose[
    PolyPlot[Θ[TorusKnot @@ #], ImageSize → 480],
    TubePlot[TorusKnot @@ #, ImageSize → 240],
    {Right, Bottom}, {Right, Bottom}
   ] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}]
```

Out[∘]=
*pdf*



## The Rank 2 Formulas

*exec*

```
nb2tex$PDFWidth = 4.625
```

*pdf:LX*

In[∘]:=

$$\mathcal{L}[X_{i\_,j\_}[s\_]] := T_3^s\,\mathbb{E}\Big[\text{CF@Plus}\Big[$$
$$\sum_{\nu=1}^{3}\left(x_{\nu i}\,(p_{\nu i^+}-p_{\nu i}) + x_{\nu j}\,(p_{\nu j^+}-p_{\nu j}) + \left(T_\nu^s-1\right) x_{\nu i}\,(p_{\nu i^+}-p_{\nu j^+})\right),$$
$$\left(T_1^s-1\right) p_{3j}\,x_{1i}\left(T_2^s\,x_{2i}-x_{2j}\right),$$
$$\epsilon\, s\left(T_3^s-1\right) p_{1j}\,(p_{2i}-p_{2j})\,x_{3i}\Big/\left(T_2^s-1\right),$$
$$\epsilon\, s\Big(1/2 + T_2^s\,p_{1i}\,p_{2j}\,x_{1i}\,x_{2i} - p_{1i}\,p_{2j}\,x_{1i}\,x_{2j} - p_{3i}\,x_{3i} - \left(T_2^s-1\right) p_{2j}\,p_{3i}\,x_{2i}\,x_{3i} +$$
$$\left(T_3^s-1\right) p_{2j}\,p_{3j}\,x_{2i}\,x_{3i} + 2\,p_{2j}\,p_{3i}\,x_{2j}\,x_{3i} + p_{1i}\,p_{3j}\,x_{1i}\,x_{3j} - p_{2i}\,p_{3j}\,x_{2i}\,x_{3j} - T_2^s\,p_{2j}\,p_{3j}\,x_{2i}\,x_{3j} +$$
$$\Big(\left(T_1^s-1\right) p_{1j}\,x_{1i}\left(T_2^{2s}\,p_{2j}\,x_{2i} - T_2^s\,p_{2j}\,x_{2j} - \left(T_2^s+1\right)\left(T_3^s-1\right) p_{3j}\,x_{3i} + T_2^s\,p_{3j}\,x_{3j}\right) +$$
$$\left(T_3^s-1\right) p_{3j}\,x_{3i}\left(1 - T_2^s\,p_{1i}\,x_{1i} + p_{2i}\,x_{2j} + \left(T_2^s-2\right) p_{2j}\,x_{2j}\right)\Big)\Big/\left(T_2^s-1\right)\Big)\Big]\Big]$$

*pdf:LC*

In[∘]:=

$$\mathcal{L}[C_{i\_}[\varphi\_]] := T_3^{\varphi}\,\mathbb{E}\left[\sum_{\nu=1}^{3} x_{\nu i}\,(p_{\nu i^+}-p_{\nu i}) + \epsilon\,\varphi\,(p_{3i}\,x_{3i} - 1/2)\right]$$