Chern-Simons-Witten Theory Near The Co-Commutative Limit

Abstract. Perhaps every algebra meeting should have one analysis talk (and vice versa), lest we forget that the other exists. In my role as the outsider, I will tell you today about the other – perturbative – evaluation of path integrals, where instead of hoping that nature will help us compute faster, we approximate nature by things we already can compute quickly.

Specifically I will tell you how in the Chern-Simons-Witten theory you can perturb the base Lie algebra from where it's easy towards where it's strong, leading to the strongest genuinely computable knot invariant we presently have.

I wish I could give my talk in the language of the Kabbalah, but I ain't smart enough for that. So I'll highlight the Kabbalistic points that we're still missing, and then stick to the Talmud.

van der Veen

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Talmud vs Kabbalah

AI Overview

The Talmud is the foundational, comprehensive compilation of Jewish law (Halacha) and legal debates, focusing on the "how" of daily practice. Kabbalah is the esoteric, mystical tradition of Judaism, focusing on the inner meaning of the Torah, the nature of God, and spiritual universes. Both are considered part of the Oral Law tradition.

kabbalah age limit

Traditionally, it was recommended to wait until age 40 to study **Kabbalah**. This guideline was meant for men to first master Jewish law and Talmud, ensuring maturity.

Compliance Statement. This talk touches *speculative* Kabbalah. I'm over 40, and it's speculation with Talmudic results. I merely speculate on where they may be coming from.

The Overall Map.

Chern-Simons-Witten with metrized Lie algebra $\mathfrak{g}$ and representation V:

$$\int_{A\in\Omega^1(\mathbb{R}^3;\mathfrak{g})}\mathcal{D}A\, e^{\frac{i}{4\pi}\int_{\mathbb{R}^3}\operatorname{tr}\left(A\wedge dA+\frac{2\sqrt{\hbar}}{3}A\wedge A\wedge A\right)}\operatorname{tr}_V\mathcal{P}\exp_\gamma(\sqrt{\hbar}A)$$

CFT, other magic

Insufficently understood?

Old Kabbalah: perturbation theory, Feynman diagrams, configuration space integrals, the KZ connection.

Old Kabbalah

Old Talmud

Reshetikhin-Turaev for $(\mathfrak{g}, V)$ at $q=e^\hbar$

$\hbar$-expansion (B-N, Lin, ...)

Finite type invariants and the Kontsevich Integral: Very powerful, but except for low cases, impossible to compute.

Old

New

The co-commutative limit: replace $\mathfrak{g}$ by $\mathfrak{g}_\epsilon$. "Solvable Approximation".

Chern-Simons-Witten with $\mathfrak{g}_\epsilon$:

$$\int_{A\in\Omega^1(\mathbb{R}^3;\mathfrak{g}_\epsilon)}\mathcal{D}A\, e^{\frac{i}{4\pi}\int_{\mathbb{R}^3}\operatorname{tr}\left(A\wedge dA+\frac{2\sqrt{\hbar}}{3}A\wedge A\wedge A\right)}\mathcal{P}\exp_\gamma(\sqrt{\hbar}A)$$

Variants of the old tricks?

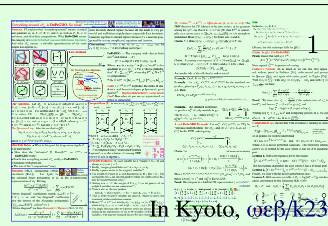
New Kabbalah (missing):
1. Exact evaluation at $\epsilon=0$ giving Δ.
2. Then perturbation theory.

New Kabbalah

New Talmud

Universal Reshetikhin-Turaev for $\mathfrak{g}_\epsilon$ with $q=e^{\hbar\epsilon}$

ϵ-expansion: perturbation theory, Feynman diagrams

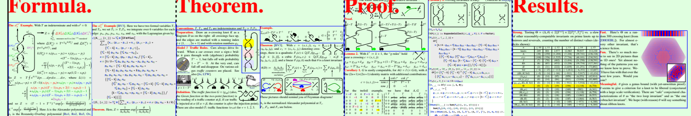


In Kyoto, ωεβ/k23

Today's invariants:

$$\begin{pmatrix} \epsilon=0 & \epsilon^2=0 & \epsilon^3=0 & \dots \\ \text{Alexander's }\Delta & \rho_1,\theta & \rho_2,\dots & \dots \end{pmatrix}$$

Formula. **Theorem.** **Proof.** **Results.**

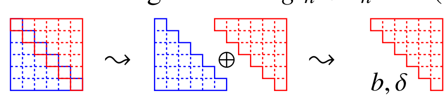


Gaussian Integration / Perturbation Theory / Feynman Diagrams. With Einstein sumintegration and $G=A^{-1}=(a_{ij})^{-1}$,

$$\int_{x\in\mathbb{R}^M} dx\, e^{-a^{ij}x_ix_j/2+\epsilon V(x)} = \frac{1}{\det^{1/2}(A)}\, e^{g_{ij}\partial_{x_i}\partial_{x_j}} e^{\epsilon V(x)}\Big|_{x=0}.$$

• Kabbalah: If M is continuous, a reduction of path integrals to finite dimensional integration. • Talmud: If M is discrete, approximates an exponential-time computation with a polynomial-time one! • Cements my status as an outsider here...

Co-Commutative Limit / Solvable Approximation. In gl_n, half is enough! Indeed $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(\lnot,b,\delta)$:



$b(\lnot)=b\colon \lnot\otimes\lnot\to\lnot$

$b(\triangle)\rightsquigarrow\delta\colon \lnot\to\lnot\otimes\lnot$

Now define $gl_n^\epsilon := \mathcal{D}(\lnot,b,\epsilon\delta)$. Schematically, this is $[\lnot,\lnot]=\lnot$, $[\triangle,\triangle]=\epsilon\triangle$, and $[\lnot,\triangle]=\triangle+\epsilon\lnot$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1}=0$ always yields a solvable Lie algebra.

Why Believe?