Pensieve header: Notebook for DBN Lecture 3 in Matemale, April 2018. See http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/.

Today. First Dror, then Roland.

Unrelated Question. Anybody wants a French (azerty) keyboard for a surface pro?

Before we Start...

Please think about our Partial To do List!

A Partial To Do List.

- Complete all "docility" arguments by identifying a "contained" docile substructure.
- Understand denominators and get rid of them.
- See if much can be gained by including P in the exponential: $\mathbb{e}^{L+Q}P \rightsquigarrow \mathbb{e}^{L+Q+P}$?
- Clean the program and make it efficient.
- Run it for all small knots and links, at $k = 2, 3$.
- Understand the centre and figure out how to read the output.
- Execute the Drinfel'd double procedure at $\mathbb{E}$-level (and thus get rid of `DeclareAlgebra` and all that is around it!).
- Extend to sl_3 and beyond.
- Do everything with `Zip` and `Bind` as the fundamentals, without ever referring back to (quantized) Lie algebras.
- Prove a genus bound and a Seifert formula.
- Obtain "Gauss-Gassner formulas" (ωεβ/NCSU).
- Relate with Melvin-Morton-Rozansky and with Rozansky-Overbay.
- Understand the braid group representations that arise.
- Find a topological interpretation. The Garoufalidis-Rozansky "loop expansion" [GR]?
- Figure out the action of the Cartan automorphism.
- Disprove the ribbon-slice conjecture!
- Figure out the action of the Weyl group.
- Do everything at the "arrow diagram" level of finite-type invariants of (rotational) virtual tangles.
- What else can you do with the "solvable approximations"?
- And with the "Gaussian zip and bind" technology?

Recall the Zipping Formula...

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{\eta^i z_i + y_j\zeta^j}\right\rangle_{(\zeta^j)} = \left\langle P(z_i+y_i,\zeta^j)\mathbb{e}^{\eta^i(z_i+y_i)}\right\rangle_{(\zeta^j)},$$

(proof: replace $y_j \to \hbar y_j$ and test at $\hbar = 0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^i z_i + y_j\zeta^j + q^i_j z_i\zeta^j}\right\rangle_{(\zeta^j)} = \det(\tilde{q})\left\langle P(\tilde{q}^k_i(z_k+y_k),\zeta^j)\mathbb{e}^{c+\eta^i\tilde{q}^k_i(z_k+y_k)}\right\rangle_{(\zeta^j)}$$

where $\tilde{q}$ is the inverse matrix of $1-q$: $(\delta^i_j - q^i_j)\tilde{q}^j_k = \delta^i_k$ (proof: replace $q^i_j \to \hbar q^i_j$ and test at $\hbar = 0$ and at $\partial_\hbar$).

... and that we only need to compute operations on exponentials...

Proposition. If $F\colon \mathcal{S}(B) \to \mathcal{S}(B')$ is linear and "continuous", then ${}^tF = \exp\left(\sum_{z_i\in B}\zeta_i z_i\right) /\!/ F$.

Our Algebra QU

QU = $\langle y, a, x, t\rangle$, with t central and subject to the relations $[a, x] = \gamma x$, $[a, y] = -\gamma y$, and $x y - q y x = \frac{1-T A^2}{\hbar}$, where $q = e^{\hbar \gamma \epsilon}$, $T = e^{\hbar t}$ ($t = \epsilon a - \gamma b$, as Roland derived the algebra), and $A = e^{-\hbar \epsilon a}$. For convenience we set $\mathcal{A} = e^{\gamma \alpha}$.

QU has a co-product Δ and an antipode S given by

$$\Delta(y,a,x,t) = (y_1 + T_1 y_2 A_1,\ a_1 + a_2,\ x_1 + A_1 x_2)$$

$$S(y,a,x,t) = (-T^{-1} A^{-1} y,\ -a,\ -A^{-1} x,\ -t)$$

The Program

Initialization / Utilities

DeclareAlgebra

DeclareMorphism

Meta-Operations

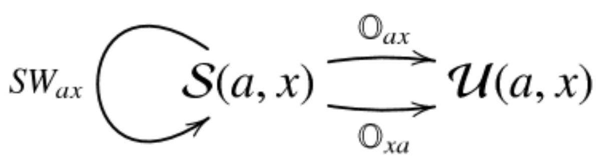
Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

The representation ρ

tSW

The 2D Lie Algebra. Clever people know* that if $[a, x] = \gamma x$ then $e^{\xi x} e^{\alpha a} = e^{\alpha a} e^{e^{-\gamma\alpha}\xi x}$. Ergo with



we have ${}^tSW_{ax} = e^{\alpha a + e^{-\gamma\alpha}\xi x}$.

$*$ Indeed $xa = (a-\gamma)x$ thus $xa^n = (a-\gamma)^n x$ thus $x e^{\alpha a} = e^{\alpha(a-\gamma)} x = e^{-\gamma\alpha} e^{\alpha a} x$ thus $x^n e^{\alpha a} = e^{\alpha a}(e^{-\gamma\alpha})^n x^n$ thus $e^{\xi x} e^{\alpha a} = e^{\alpha a} e^{e^{-\gamma\alpha}\xi x}$.

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta\left(e^{-\eta y} e^{\eta G}\right) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

In[ ]:=
```
SW_xy[U_, kk_] :=
  SW_xy[U, kk] = Block[{$U = U, $k = kk, $p = kk}, Module[{G, F, fs, f, bs, e, b, es},
     G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
     fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, l}, {j, 0, l}, {k, 0, l}];
     F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k});
     es = Flatten[
       Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}}, {b, bs}]];
     F = F /. DSolve[es, fs, η][[1]];
     𝔼[0,
       ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
       F + O_$k /. {𝕖- → 1, U → Times}
      ] /. (v : η | ξ | t | T | y | a | x) → v_1
    ]];
tSW_{xy_,i_,j_→k_} := SW_xy[$U, $k] /. {ξ_1 → ξ_i, η_1 → η_j, (v : t | T | y | a | x)_1 → v_k};
tSW_{xa,i_,j_→k_} := 𝔼[α_j a_k, 𝕖^(-γ α_j) ξ_i x_k, 1];
tSW_{ay,i_,j_→k_} := 𝔼[α_i a_k, 𝕖^(-γ α_i) η_j y_k, 1];
```

Exponentials as needed.

Task. Define $\mathrm{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi\,\mathbb{O}(P)}$ to ϵ^k in the algebra U_i, where ξ is a scalar, X is x_i or y_i, and P is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form. Should satisfy $U @ \mathrm{Exp}_{U_i,k}[\xi, P] == \mathbb{S}_U\left[e^{\xi x},\ x \to \mathbb{O}(P)\right]$.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\xi\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)$, then $F(\xi = 0) = 1$ and we have:

$$\mathbb{O}\left(e^{\xi P_0}(P_0 F(\xi) + \partial_\xi F\right) = \mathbb{O}\left(\partial_\xi e^{\xi P_0} F(\xi)\right) = \partial_\xi \mathbb{O}\left(e^{\xi P_0} F(\xi)\right) = \partial_\xi e^{\xi\,\mathbb{O}(P)} = e^{\xi\,\mathbb{O}(P)}\, \mathbb{O}(P) = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right) \mathbb{O}(P).$$

This is an ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ.

In[ ]:=
```
(* Bug: The first line is valid only if O(𝕖^P0) == 𝕖^O(P0). *)
(* Bug: ξ must be a symbol. *)
Exp_{U_i_,0}[ξ_, P_] := Module[{LQ = Normal@P /. ϵ → 0},
   𝔼[ξ LQ /. (x | y)_i → 0, ξ LQ /. (t | a)_i → 0, 1] ];
Exp_{U_i_,k_}[ξ_, P_] := Block[{$U = U, $k = k},
  Module[{P0, φ, φs, F, j, rhs, at0, atξ},
   P0 = Normal@P /. ϵ → 0;
   φs =
    Flatten@Table[φ_{j1,j2,j3}[ξ], {j2, 0, k}, {j1, 0, 2 k + 1 - j2}, {j3, 0, 2 k + 1 - j2 - j1}];
   F = Normal@Last@Exp_{U_i,k-1}[ξ, P] + ϵ^k φs.(φs /. φ_{js__}[ξ] :> Times @@ {y_i, a_i, x_i}^{js});
   rhs = Normal@
     Last@m_{i,j→i}[𝔼[ξ P0 /. (x | y)_i → 0, ξ P0 /. (t | a)_i → 0, F + O_k] m_{i→j}@𝔼[0, 0, P + O_k]];
   at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. ξ → 0, {y_i, a_i, x_i}];
   atξ = (# == 0) & /@ Flatten@CoefficientList[(∂_ξ F) + P0 F - rhs, {y_i, a_i, x_i}];
   𝔼[ξ P0 /. (x | y)_i → 0, ξ P0 /. (t | a)_i → 0, F + O_k] /.
    DSolve[And @@ (at0 ⋃ atξ), φs, ξ][[1]] ]]
```

Zip and Bind

Tensorial Representations

Alternative Algorithms

New Utilities (not on handout)

In[∘]:= $\alpha 2\mathcal{A} = \{e^{c\_ \cdot \alpha_{i\_}+b\_ \cdot} :\to \mathcal{A}_i^{c/\gamma} e^b,\ e^{c\_ \cdot \alpha+b\_ \cdot} :\to \mathcal{A}^{c/\gamma} e^b,\ e^{\mathcal{E}\_} :\to e^{\text{Expand@}\mathcal{E}}\};$

Some Runs

The Structure Tensors

In[∘]:= `$k = 1; γ = 1;`

In[∘]:= $tm_{1,2\to 3}$ //. $\alpha 2\mathcal{A}$

Out[∘]= $$\mathbb{E}\Big[a_3\,\alpha_1 + a_3\,\alpha_2 + t_3\,(\tau_1+\tau_2),\ y_3\,\eta_1 + \frac{y_3\,\eta_2}{\mathcal{A}_1} + \frac{x_3\,\xi_1}{\mathcal{A}_2} + \frac{(1-T_3)\,\eta_2\,\xi_1}{\hbar} + x_3\,\xi_2,\ 1 + \frac{1}{4\hbar}\eta_2\,\xi_1\Big(8\hbar\,a_3\,T_3 + \frac{4\hbar^2\,x_3\,y_3}{\mathcal{A}_1\mathcal{A}_2} + \frac{2\hbar\,y_3\,\eta_2}{\mathcal{A}_1} - \frac{6\hbar\,T_3\,y_3\,\eta_2}{\mathcal{A}_1} + \frac{2\hbar\,x_3\,\xi_1}{\mathcal{A}_2} - \frac{6\hbar\,T_3\,x_3\,\xi_1}{\mathcal{A}_2} + \eta_2\,\xi_1 - 4\,T_3\,\eta_2\,\xi_1 + 3\,T_3^2\,\eta_2\,\xi_1\Big)\,\epsilon + O[\epsilon]^2\Big]$$

But that’s not the form we like! Let’s open Zip and Bind!

How was it computed? Let’s open the tSW box!

In[∘]:= tS_1

Out[∘]= $$\mathbb{E}\Big[-a_1\,\alpha_1 - t_1\,\tau_1,\ \frac{1}{\hbar\,T_1}\big(-e^{\alpha_1}\hbar\,y_1\,\eta_1 - e^{\alpha_1}\hbar\,T_1\,x_1\,\xi_1 + e^{\alpha_1}\eta_1\,\xi_1 - e^{\alpha_1}T_1\,\eta_1\,\xi_1\big),\ 1 + \frac{1}{4\hbar\,T_1^2}\big(4e^{\alpha_1}\hbar^2\,T_1\,y_1\,\eta_1 - 4e^{\alpha_1}\hbar^2\,a_1\,T_1\,y_1\,\eta_1 - 2e^{2\alpha_1}\hbar^2\,y_1^2\,\eta_1^2 - 4e^{\alpha_1}\hbar^2\,a_1\,T_1^2\,x_1\,\xi_1 - 4e^{\alpha_1}\hbar\,T_1\,\eta_1\,\xi_1 + 8e^{\alpha_1}\hbar\,a_1\,T_1\,\eta_1\,\xi_1 + 4e^{\alpha_1}\hbar\,T_1^2\,\eta_1\,\xi_1 - 4e^{2\alpha_1}\hbar^2\,T_1\,x_1\,y_1\,\eta_1\,\xi_1 + 6e^{2\alpha_1}\hbar\,y_1\,\eta_1^2\,\xi_1 - 2e^{2\alpha_1}\hbar\,T_1\,y_1\,\eta_1^2\,\xi_1 - 2e^{2\alpha_1}\hbar^2\,T_1^2\,x_1^2\,\xi_1^2 + 6e^{2\alpha_1}\hbar\,T_1\,x_1\,\eta_1\,\xi_1^2 - 2e^{2\alpha_1}\hbar\,T_1^2\,x_1\,\eta_1\,\xi_1^2 - 3e^{2\alpha_1}\eta_1^2\,\xi_1^2 + 4e^{2\alpha_1}T_1\,\eta_1^2\,\xi_1^2 - e^{2\alpha_1}T_1^2\,\eta_1^2\,\xi_1^2\big)\,\epsilon + O[\epsilon]^2\Big]$$

How was it computed? Let’s open Exponentials as Needed!

In[∘]:= $t\Delta_{1\to 1,2}$

Out[∘]= $$\mathbb{E}\Big[(a_1+a_2)\,\alpha_1 + (t_1+t_2)\,\tau_1,\ y_1\,\eta_1 + T_1\,y_2\,\eta_1 + x_1\,\xi_1 + x_2\,\xi_1,\ 1 + \frac{1}{2}\big(-2\hbar\,a_1\,T_1\,y_2\,\eta_1 + \hbar\,T_1\,y_1\,y_2\,\eta_1^2 - 2\hbar\,a_1\,x_2\,\xi_1 + \hbar\,x_1\,x_2\,\xi_1^2\big)\,\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= $\{tR_{1,2},\ \overline{tR}_{1,2}\}$

Out[∘]= $$\left\{\mathbb{E}\left[-\hbar\, a_2\, t_1,\ \hbar\, x_2\, y_1,\ 1+\left(\hbar\, a_1\, a_2-\frac{1}{4}\hbar^3\, x_2^2\, y_1^2\right)\epsilon+O[\epsilon]^2\right],\ \mathbb{E}\left[\hbar\, a_2\, t_1,\ -\frac{\hbar\, x_2\, y_1}{T_1},\ 1+\frac{1}{4\,T_1^2}\left(-4\,\hbar\, a_1\, a_2\, T_1^2-4\,\hbar^2\, a_1\, T_1\, x_2\, y_1-4\,\hbar^2\, a_2\, T_1\, x_2\, y_1-3\,\hbar^3\, x_2^2\, y_1^2\right)\epsilon+O[\epsilon]^2\right]\right\}$$

In[∘]:= $\{tC_1,\ \overline{tC}_2\}$

Out[∘]= $$\left\{\mathbb{E}\left[0,\ 0,\ \sqrt{T_1}-\hbar\, a_1\sqrt{T_1}\,\epsilon+O[\epsilon]^2\right],\ \mathbb{E}\left[0,\ 0,\ \frac{1}{\sqrt{T_2}}+\frac{\hbar\, a_2\,\epsilon}{\sqrt{T_2}}+O[\epsilon]^2\right]\right\}$$

Some Testing

Associativity of tm.

In[∘]:= $tm_{1,2\to 2}\sim B_2\sim tm_{2,3\to 1}$

Out[∘]= $$\mathbb{E}\Big[a_1\,\alpha_1+a_1\,\alpha_2+a_1\,\alpha_3+t_1\,\tau_1+t_1\,\tau_2+t_1\,\tau_3,$$
$$\frac{1}{\hbar}e^{-\alpha_1-\alpha_2-\alpha_3}\big(e^{\alpha_1+\alpha_2+\alpha_3}\hbar\, y_1\,\eta_1+e^{\alpha_2+\alpha_3}\hbar\, y_1\,\eta_2+e^{\alpha_3}\hbar\, y_1\,\eta_3+e^{\alpha_1}\hbar\, x_1\,\xi_1+e^{\alpha_1+\alpha_2+\alpha_3}\eta_2\,\xi_1-e^{\alpha_1+\alpha_2+\alpha_3}T_1\,\eta_2\,\xi_1+$$
$$e^{\alpha_1+\alpha_3}\eta_3\,\xi_1-e^{\alpha_1+\alpha_3}T_1\,\eta_3\,\xi_1+e^{\alpha_1+\alpha_2}\hbar\, x_1\,\xi_2+e^{\alpha_1+\alpha_2+\alpha_3}\eta_3\,\xi_2-e^{\alpha_1+\alpha_2+\alpha_3}T_1\,\eta_3\,\xi_2+e^{\alpha_1+\alpha_2+\alpha_3}\hbar\, x_1\,\xi_3\big),$$
$$1+\frac{1}{4\,\hbar}e^{-\alpha_1-2\,\alpha_2-\alpha_3}\big(8\,e^{\alpha_1+2\,\alpha_2+\alpha_3}\hbar\, a_1\,T_1\,\eta_2\,\xi_1+4\,e^{\alpha_2}\hbar^2\,x_1\,y_1\,\eta_2\,\xi_1+2\,e^{2\,\alpha_2+\alpha_3}\hbar\, y_1\,\eta_2^2\,\xi_1-$$
$$6\,e^{2\,\alpha_2+\alpha_3}\hbar\,T_1\,y_1\,\eta_2^2\,\xi_1+8\,e^{\alpha_1+\alpha_2+\alpha_3}\hbar\,a_1\,T_1\,\eta_3\,\xi_1+4\,\hbar^2\,x_1\,y_1\,\eta_3\,\xi_1+4\,e^{\alpha_2+\alpha_3}\hbar\,y_1\,\eta_2\,\eta_3\,\xi_1-$$
$$12\,e^{\alpha_2+\alpha_3}\hbar\,T_1\,y_1\,\eta_2\,\eta_3\,\xi_1+2\,e^{\alpha_3}\hbar\,y_1\,\eta_3^2\,\xi_1-6\,e^{\alpha_3}\hbar\,T_1\,y_1\,\eta_3^2\,\xi_1+2\,e^{\alpha_1+\alpha_2}\hbar\,x_1\,\eta_2\,\xi_1^2-$$
$$6\,e^{\alpha_1+\alpha_2}\hbar\,T_1\,x_1\,\eta_2\,\xi_1^2+e^{\alpha_1+2\,\alpha_2+\alpha_3}\eta_2^2\,\xi_1^2-4\,e^{\alpha_1+2\,\alpha_2+\alpha_3}T_1\,\eta_2^2\,\xi_1^2+3\,e^{\alpha_1+2\,\alpha_2+\alpha_3}T_1^2\,\eta_2^2\,\xi_1^2+2\,e^{\alpha_1}\hbar\,x_1\,\eta_3\,\xi_1^2-$$
$$6\,e^{\alpha_1}\hbar\,T_1\,x_1\,\eta_3\,\xi_1^2+2\,e^{\alpha_1+\alpha_2+\alpha_3}\eta_2\,\eta_3\,\xi_1^2-8\,e^{\alpha_1+\alpha_2+\alpha_3}T_1\,\eta_2\,\eta_3\,\xi_1^2+6\,e^{\alpha_1+\alpha_2+\alpha_3}T_1^2\,\eta_2\,\eta_3\,\xi_1^2+$$
$$e^{\alpha_1+\alpha_3}\eta_3^2\,\xi_1^2-4\,e^{\alpha_1+\alpha_3}T_1\,\eta_3^2\,\xi_1^2+3\,e^{\alpha_1+\alpha_3}T_1^2\,\eta_3^2\,\xi_1^2+8\,e^{\alpha_1+2\,\alpha_2+\alpha_3}\hbar\,a_1\,T_1\,\eta_3\,\xi_2+4\,e^{\alpha_2}\hbar^2\,x_1\,y_1\,\eta_3\,\xi_2+$$
$$2\,e^{\alpha_2+\alpha_3}\hbar\,y_1\,\eta_3^2\,\xi_2-6\,e^{\alpha_2+\alpha_3}\hbar\,T_1\,y_1\,\eta_3^2\,\xi_2+4\,e^{\alpha_1+\alpha_2}\hbar\,x_1\,\eta_3\,\xi_1\,\xi_2-12\,e^{\alpha_1+\alpha_2}\hbar\,T_1\,x_1\,\eta_3\,\xi_1\,\xi_2+$$
$$2\,e^{\alpha_1+\alpha_2+\alpha_3}\eta_3^2\,\xi_1\,\xi_2-8\,e^{\alpha_1+\alpha_2+\alpha_3}T_1\,\eta_3^2\,\xi_1\,\xi_2+6\,e^{\alpha_1+\alpha_2+\alpha_3}T_1^2\,\eta_3^2\,\xi_1\,\xi_2+2\,e^{\alpha_1+2\,\alpha_2}\hbar\,x_1\,\eta_3\,\xi_2^2-$$
$$6\,e^{\alpha_1+2\,\alpha_2}\hbar\,T_1\,x_1\,\eta_3\,\xi_2^2+e^{\alpha_1+2\,\alpha_2+\alpha_3}\eta_3^2\,\xi_2^2-4\,e^{\alpha_1+2\,\alpha_2+\alpha_3}T_1\,\eta_3^2\,\xi_2^2+3\,e^{\alpha_1+2\,\alpha_2+\alpha_3}T_1^2\,\eta_3^2\,\xi_2^2\big)\,\epsilon+O[\epsilon]^2\Big]$$

In[∘]:= $tm_{1,2\to fran}\sim B_{fran}\sim tm_{fran,3\to 1}\equiv tm_{2,3\to 2}\sim B_2\sim tm_{1,2\to 1}$

Out[∘]= True

tS is an anti-homomorphism for tm.

In[∘]:= $(tS_1\ tS_2)\sim B_{1,2}\sim tm_{1,2\to 1}\equiv tm_{2,1\to 1}\sim B_1\sim tS_1$

Out[∘]= True

Testing convolution inverse:

In[∘]:= $t\Delta_{1\to 1,2}\sim B_1\sim tS_1\sim B_{1,2}\sim tm_{1,2\to 1}$

Out[∘]= $\mathbb{E}\left[0,\ 0,\ 1+O[\epsilon]^2\right]$

Testing quasi-triangular axioms

In[∘]:= $(t\Delta_{1\text{->}1,2}\ tR_{3,4})\sim B_{1,2,3,4}\sim (tm_{1,3\to 1}\ tm_{2,4\to 2})\equiv (t\Delta_{1\to 2,1}\ tR_{3,4})\sim B_{1,2,3,4}\sim (tm_{3,1\to 1}\ tm_{4,2\to 2})$

Out[∘]= True

Testing R3

In[∘]:= $\left(\mathrm{tR}_{2,3}\,\mathrm{tR}_{1,4}\,\mathrm{tR}_{5,6}\right) \sim \mathrm{B}_{1,2,3,4,5,6} \sim \left(\mathrm{tm}_{1,5\to 1}\,\mathrm{tm}_{2,6\to 2}\,\mathrm{tm}_{3,4\to 3}\right) \equiv \left(\mathrm{tR}_{1,2}\,\mathrm{tR}_{5,3}\,\mathrm{tR}_{6,4}\right) \sim \mathrm{B}_{1,2,3,4,5,6} \sim \left(\mathrm{tm}_{1,5\to 1}\,\mathrm{tm}_{2,6\to 2}\,\mathrm{tm}_{3,4\to 3}\right)$

Out[∘]= True

Docility and why it matters:

Definition. A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form

doc·ile
/ˈdäsəl/
adjective
ready to accept control or instruction; submissive
"a cheap and docile workforce"

$$e^{q^{ij}z_iz_j}P = e^{q^{ij}z_iz_j}\left(\sum_{k\geq 0}\epsilon^k P_k\right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k \leq 4k$.

Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$.

In our case our invariants and operations are of the form $e^{L+Q}P$, where

- L is a quadratic of the form $\sum l_{z\zeta} z\zeta$, where z runs over $\{t_i, \alpha_i\}_{i\in S}$ and ζ runs over $\{\tau_i, a_i\}_{i\in S}$, with integer coefficients $l_{z\zeta}$.
- Q is a quadratic of the form $\sum q_{z\zeta} z\zeta$, where z runs over $\{x_i, \eta_i\}_{i\in S}$ and ζ runs over $\{\xi_i, y_i\}_{i\in S}$, with coefficients $q_{z\zeta}$ in the ring R_s of rational functions in $(T_i)_{i\in S}$ and $(\mathcal{A}_i)_{i\in S}$.
- $P = \sum \epsilon^k P_k$ is a docile power series in $\{y_i, a_i, x_i, \eta_i, \xi_i\}_{i\in S}$, where $\deg(y_i, a_i, x_i, \eta_i, \xi_i) = (1, 2, 1, 1, 1)$.

The Trefoil

Seeing that our invariant is poly-time and the trefoil knot is tiny, the following will compute in no time at all:

In[∘]:= $\mathrm{Timing@Block}\big[\{\$k = 1\},$
$\quad Z = \mathrm{tR}_{1,5}\,\mathrm{tR}_{6,2}\,\mathrm{tR}_{3,7}\,\overline{\mathrm{tC}_4}\,\overline{\mathrm{tKink}_8}\,\overline{\mathrm{tKink}_9}\,\overline{\mathrm{tKink}_{10}};$
$\quad \mathrm{Do}[Z = Z \sim \mathrm{B}_{1,k} \sim \mathrm{tm}_{1,k\to 1},\ \{k, 2, 10\}];\ Z\big]$

Out[∘]= $\Big\{108.703,$
$\mathbb{E}\Big[0,\ 0,\ \frac{T_1}{1 - T_1 + T_1^2} + \big(\big(-2\hbar\, a_1 T_1 - \hbar T_1^2 + 2\hbar\, a_1 T_1^2 + 2\hbar T_1^3 - 3\hbar T_1^4 - 2\hbar\, a_1 T_1^4 + 2\hbar T_1^5 + 2\hbar\, a_1 T_1^5 - 2\hbar^2 T_1 x_1 y_1 - 2\hbar^2 T_1^4 x_1 y_1\big)\,\epsilon\big) \big/ \big(1 - 3T_1 + 6T_1^2 - 7T_1^3 + 6T_1^4 - 3T_1^5 + T_1^6\big) + O[\epsilon]^2\Big]\Big\}$

In[ ]:=
```
Timing@Block[{$k = 1},
  Z = tR1,5 tR6,2 tR3,7 tC̄4 tKink̄8 tKink̄9 tKink̄10 /. T_ → T1;
  Do[Z = Z ~ B1,k ~ tm1,k→1 /. T_ → T1, {k, 2, 10}]; Z]
```

Out[ ]=

$$\left\{4.875, \mathbb{E}\left[0, 0, \frac{T_1}{1 - T_1 + T_1^2} + \left(\left(-2\hbar a_1 T_1 - \hbar T_1^2 + 2\hbar a_1 T_1^2 + 2\hbar T_1^3 - 3\hbar T_1^4 - 2\hbar a_1 T_1^4 + 2\hbar T_1^5 + 2\hbar a_1 T_1^5 - 2\hbar^2 T_1 x_1 y_1 - 2\hbar^2 T_1^4 x_1 y_1\right)\epsilon\right) \Big/ \left(1 - 3T_1 + 6T_1^2 - 7T_1^3 + 6T_1^4 - 3T_1^5 + T_1^6\right) + O[\epsilon]^2\right]\right\}$$