Hopf algebra and snarl diagrams

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1 Snarls and Hopf algebras

In computing with knots it is useful to be able to freely cut a diagram of the knot into pieces. Not just pieces that fit together like square tiles as is done in the tangle category. We allow ourselves pieces that lie in arbitrary regions in the plane. Strictly speaking such pieces may only be combined when their ends fit together properly. It is however more convenient not to enforce such conditions explicitly and formally allow any component of the diagram to be combined with any other. Closed components are not allowed and every component is oriented and carries a name. The only information that is retained from the planar projection is the rotation number of each component. This generalizes Morse diagrams (crossings, cups and caps) used to compute quantum invariants.

Definition 1. *A* **snarl**[1] **diagram** *is a finite set L together with a finite oriented graph $G = (V, E)$ and functions $\sigma : V \to \{\pm 1\}$ and $\rho : E \to \mathbb{Z}$. The edges E are assumed to be a disjoint union of oriented paths and each path is labelled by an element of L. Furthermore the edges around any vertex are ordered cyclically such that two adjacent edges enter and two exit each vertex that is not an endpoint of a path.*

The vertices of the graph should be viewed as the crossings and endpoints of a projection of a piece of a knot. The paths labeled by L correspond to the connected components. The map ρ keeps track of rotation numbers of the tangent vector on the edges so that our diagrams are like Morse diagrams. To build a snarl diagram from any knot diagram just make the tangent vector near each crossing point upwards and count the resulting rotation numbers of the tangent vector at each edge.

Knot diagrams may be assembled from simpler pieces by the following two operations on snarl diagrams.

Definition 2. Disjoint union: *For two snarl-diagrams G, G' with label sets L, L' the disjoint union $G \sqcup G'$ is the snarl diagram with underlying graph as indicated and label set $L \sqcup L'$. To avoid clutter we often omit the $\sqcup$ symbol and use juxtaposition instead.*
Stitching: *For $i \neq j \in L$ and $k \notin L - \{i, j\}$ define the snarl diagram $m^{ij}_k(G)$ to be the graph obtained from G by connecting the endpoint of component i to the start of component j, erasing the vertex in the middle. For the newly created edge e we define $\rho(e)$ to be the sum of the values of ρ on the edges that disappear. The newly created component is labeled k so the label set is $L - \{i, j\} \cup \{k\}$.*

Notice how any snarl diagram may be constructed using disjoint union and stitching from two types of fundamental diagrams: The positive and negative crossings $X_{ij}, \bar{X}_{ij}$ where we label the over-strand i and ρ of every edge is 0 and the diagram C^r_i, a single oriented edge labelled i with rotation number $\rho = r$. For convenience we set $C^0_i = \mathbb{1}_i$. Notice that the symbol C actually looks like the counter clockwise turning Morse diagram it represents.

It is sometimes useful to stitch many ends at the same time. For a sequence $I = (I_1, \ldots, I_n)$ of n distinct elements of L and $k \notin L - I$ define[2]

$$m^I_k = m^{I_1,I_2}_k /\!\!/ m^{k,I_3}_k /\!\!/ \ldots m^{k,I_{n-1}}_k /\!\!/ m^{k,I_n}_k$$

Even more generally if $\tau = (\tau^1, \ldots \tau^b)$ is a sequence of b such sequences whose disjoint union is L and $B = (B_1, \ldots, B_b)$ is any b-element sequence then define $m^\tau_B = m^{\tau^1}_{B_1} /\!\!/ \ldots /\!\!/ m^{\tau^b}_{B_b}$. When B is not specified we take it to be $(1, \ldots, b)$, the commas are sometimes dropped for brevity.

Snarl diagrams are meant to generalize Morse diagrams of pieces of knots. As such we should consider them up to equivalence under Reidemeister moves. Whenever part of a diagram describes intervals embedded in a disk we consider the diagram up to Reidemeister moves corresponding to isotopy in that disk. More formally we require:

[1] Dictionary entry: a knot or tangle, also a growl.
[2] In what follows $f /\!\!/ g$ means the composition $g \circ f$.

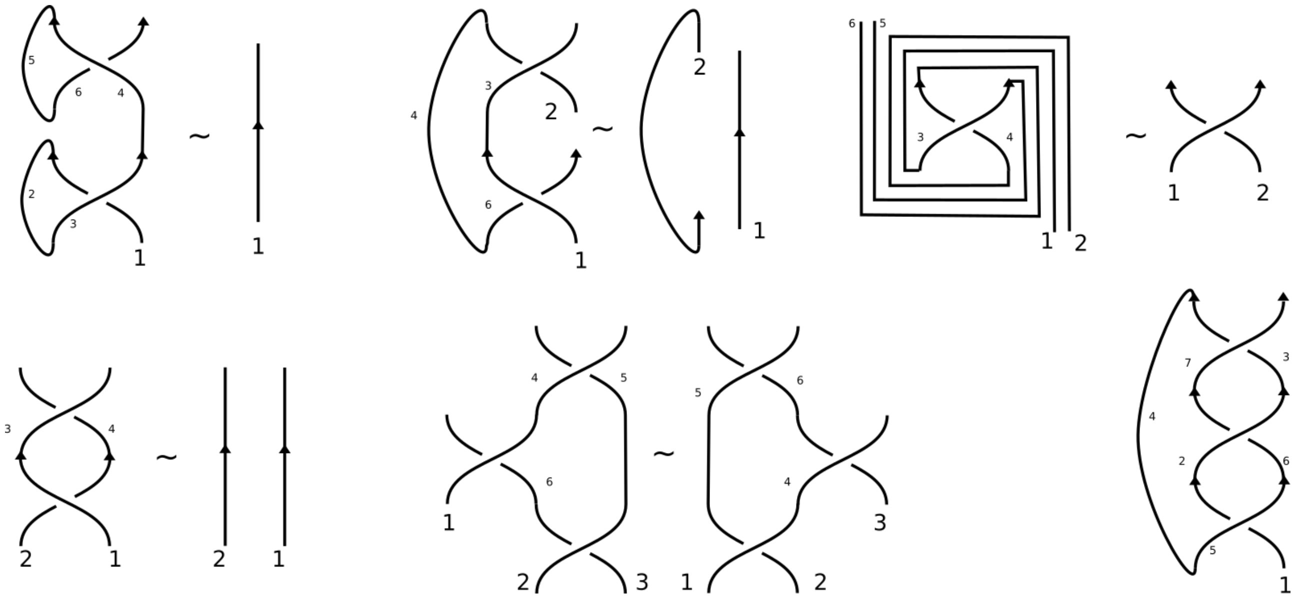

Figure 1: The Reidemeister moves for snarl diagrams plus a diagram for the trefoil.

Definition 3. *Consider the equivalence relation $\sim$ on the set of snarl diagrams generated by relabeling components and the equivalences*

$$X_{31}C_2\bar{X}_{46}C_5/\!\!/m^{(123456)} \sim \mathbb{1}_1 \qquad \textit{(framed R1)} \qquad (1)$$

$$\bar{X}_{12}X_{34}/\!\!/m^{(13)(24)} \sim \mathbb{1}_1\mathbb{1}_2 \qquad \textit{(R2)} \qquad (2)$$

$$\bar{X}_{16}X_{32}C_4/\!\!/m^{(13)(246)} \sim \mathbb{1}_1C_2 \qquad \textit{(Cyclic R2)} \qquad (3)$$

$$X_{16}X_{23}X_{45}/\!\!/m^{(14)(25)(36)} \sim X_{12}X_{43}X_{56}/\!\!/m^{(14)(25)(36)} \qquad \textit{(R3)} \qquad (4)$$

$$C_1C_2X_{34}\bar{C}_5\bar{C}_6/\!\!/m^{(135)(246)} \sim X_{12} \qquad \textit{(Swirl)} \qquad (5)$$

In the figure below each of the Reidemeister moves for snarls is illustrated. The large numbers are the labels of the components the small numbers refer to the labels of the building blocks used to construct the snarls. We also drew the diagram of a trefoil knot. $K_{3_1} = X_{51}X_{26}X_{73}C_4/\!\!/m^{1234567}$.

1.1 Hopf algebra operations

As a foreshadowing of the Hopf-algebras to come let us describe a few more operations on snarl diagrams that make topological sense.

Doubling a strand $\Delta^i_{\ell,r}$. Using the framing we may double component i and call the two resulting components ℓ, r. Walking along the orientation ℓ denotes the left-most and r the right-most strand.

Deleting a strand: η_i deletes strand i.

Unit $\mathbb{1}_i$ adds unit named i.

Reversing a strand S. To formally define this operation it suffices to say $m^{ij}_k/\!\!/S_k = S_i/\!\!/S_j/\!\!/m^{ji}_k$ and what it does on the generators X, C. Define $S_i(C^r_i) = C^r_i$ and

$$S_i(X_{ij}) = \bar{C}_{i_1}\bar{X}_{i_2j}C_{i_3}/\!\!/m^{(i_1i_2i_3)(j)}_{ij} \quad S_j(X_{ij}) = \bar{X}_{ij} \quad S_i(\bar{X}_{ij}) = X_{ij} \quad S_j(\bar{X}_{ij}) = \bar{C}_{i_1}X_{i_2j}C_{i_3}/\!\!/m^{(i_1i_2i_3)(j)}_{ij}$$

The defining equations of the tensor-algebra over a quasi-triangular ribbon Hopf algebra now all make sense in the context of snarl diagrams. Except for linearity they all hold.

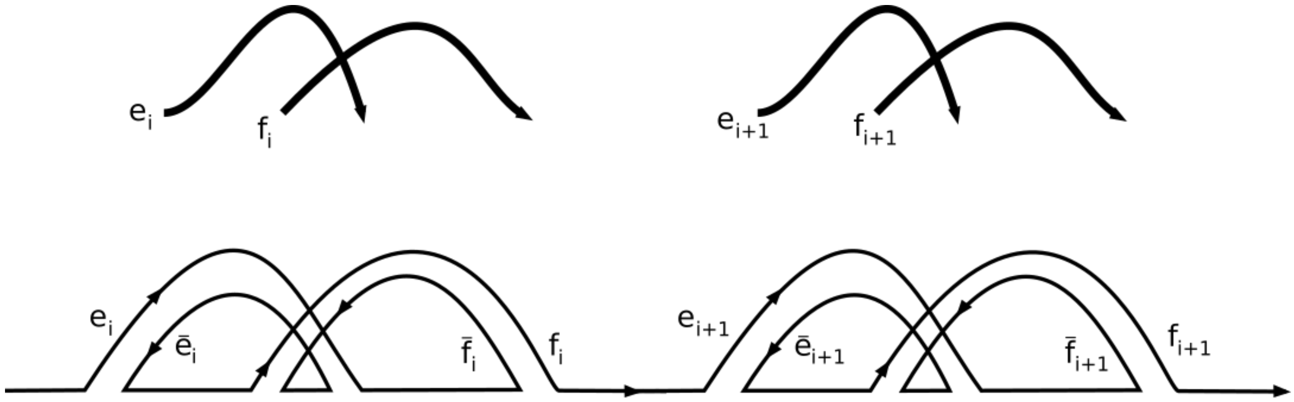

Figure 2: A caravan with two camels, or a genus two Seifert surface (bottom) created from snarl diagram T (top).

$m^{ij}_j /\!\!/ m^{jk}_i = m^{jk}_j /\!\!/ m^{ij}_i$	Associativity
$\Delta^i_{ij} /\!\!/ \Delta^j_{jk} = \Delta^i_{ik} /\!\!/ \Delta^i_{ij}$	co-Associativity
$m^{ij}_k /\!\!/ \Delta^k_{kk'} = \Delta^i_{ii'}\Delta^j_{jj'} /\!\!/ m^{(ij)(i'j')}_{kk'}$	bi-algebra
$\Delta^i_{ij} /\!\!/ S_i /\!\!/ m^{ij}_i = \eta_i /\!\!/ \mathbb{1}_i$	Convolution inverse
$X_{ik} /\!\!/ \Delta^i_{ij} = X_{ik'}X_{jk} /\!\!/ m^{k,k'}_k$	Quasi-triangular axiom 1
$\Delta^i_{ij} /\!\!/ X_{i'j'} /\!\!/ m^{(ii')(jj')}_{ij} = \Delta^i_{j'i'} /\!\!/ X_{ij} /\!\!/ m^{(ii')(jj')}_{ij}$	Quasi-triangular axiom 2
$s = X_{ij} /\!\!/ S_i /\!\!/ m^{ji}_i \quad S(s)s$	Drinfeld element and square of ribbon element

In the tensor algebra of any vector space V we use subscripts to indicate the tensor factor an element is at. For example $a_2 = 1 \otimes a$ and $R_{31} = \beta \otimes 1 \otimes \alpha$ whenever $R = \alpha \otimes \beta \in V^{\otimes 2}$. Components of a snarl thus are made to correspond to tensor factors of an algebra.

1.2 Knot genus

Knot genus may be discussed algebraically using a band-presentation of the Seifert surface. The surface may be isotoped to a disk with $2g$ bands attatched, as a caravan of camels. Starting from any snarl T with $2g$ components labelled with e_i, f_i, where $i = 1...g$ we may form the caravan as follows. Apply the following operation to T

$$/\!\!/_{i=1}^{g} \left(\Delta^{e_i}_{e_i,\bar{e}_i} \Delta^{f_i}_{f_i,\bar{f}_i} /\!\!/ S_{\bar{e}_i} /\!\!/ S_{\bar{f}_i} /\!\!/ m^{e_i \bar{f}_i \bar{e}_i f_i}_i \right) /\!\!/ m^{1,2,3...g}$$

2 The algebras in the portfolio

Definition 4. *The Hopf algebra QU depends on parameters*[3] $\epsilon, \hbar$, *is generated by* y, a, x, t *with relations*

$$[a, y] = -y \quad [x, a] = -x \quad xy - qyx = \frac{1 - TA^2}{\hbar}$$

here $q = e^{\hbar\epsilon}$ *and* $T = e^{\hbar t}$ *and* $A = e^{-\hbar\epsilon a}$ *and exponentials are understood as series in* $\hbar$.

The co-algebra structure is

$$\Delta(y) = y_1 + y_2 T_1 A_1 \quad \Delta(a) = a_1 + a_2 \quad \Delta(x) = x_1 + A_1 x_2 \quad \Delta(t) = t_1 + t_2$$

with antipodes:

$$S(y) = -T^{-1}A^{-1}y \quad S(a) = -a \quad S(x) = -A^{-1}x \quad S(t) = -t$$

and co-unit that sends all generators to 0.

[3] Notice the additional parameter γ was set to 1 here.

The Hopf algebra QU is isomorphic to the Drinfeld double of the sub-Hopf algebra $\langle a, x\rangle$. This is how one can obtain the R-matrix and spinner C:

$$R_{ij} = e^{\hbar(\epsilon a_i - t_i)a_j} e_q^{\hbar y_i x_j} \qquad C = T^{\frac{1}{2}} A$$

where $e_q^z = \sum_{k=0}^{\infty} \frac{z^k}{[k]!}$ and $[k] = \frac{1-q^k}{1-q}$.

It is a little easier to compute with the generator $Y = A^{-1}y$ because then $[x, Y] = xA^{-1}y - A^{-1}yx = q^{-1}A^{-1}(xy - qyx) = \frac{A^{-1}-TA}{q^{-1}\hbar}$.

The algebra CU may be obtained from QU by taking the 'clasical limit' $\hbar \to 0$ in all the equations. The benefit of CU is that it is a universal enveloping algebra of a Lie algebra. However the R-matrix becomes trivial in this limit.

3 Computations in QU

PBW theorem: Any element in QU may be written uniquely in yax order: $\sum_{i,j,k} d_{i,j,k}(t) y^i a^j x^k$. This allows unambiguous computations, for example $(x + a)yx = qyx^2 + \frac{1-TA^2}{\hbar}x + y(a-1)x$. To formalize this define for $B = (y, a, x)$ the completed symmetric power to be $\mathcal{S}(B)$, i.e. commutative power series in the variables B. For any PBW basis B we have a map $\mathbb{O}_B : \mathcal{S}(B) \to QU$ defined by ordering the monomials according to the basis B. By default $B = (y, a, x)$. For example $\mathbb{O}(xy) = yx$ and $\mathbb{O}(e^{xy}) = \sum_{k=0}^{\infty} \frac{y^k x^k}{k!}$.

3.1 Docile perturbed Gaussians

A docile perturbed Gaussian is an expression of the form $\mathbb{O}e^G P$ where G is a polynomial of degree ≤ 2 in y, a, x (the Gaussian) that does not depend on ϵ. The perturbation P is a power series in ϵ such that each monomial of P satisfies $\deg_x + \deg_y + \deg_a \leq 2\deg_\epsilon + 2$.

The set $\mathcal{D}$ of all docile perturbed Gaussians is the subset of QU where we do our computations. Notice both the R-matrix and the spinner C are docile. This follows from the Faddeev-Quesne formula $e_q^z = e^{\sum_{k=1}^{\infty} \frac{(1-q)^k z^k}{(1-q^k)k}}$.

We aim to prove

Theorem 1. *$\mathcal{D}$ is closed under all Hopf-algebra operations except addition.*

$\mathcal{D}_0$ is the subset of docile perturbations, i.e. those where $G = 0$. Notice that the exponential map sends $\mathcal{D}_0$ to $\mathcal{D}$. $\mathcal{D}_0$ is a vector subspace of QU. $\mathcal{D}_0$ is also closed under taking commutators.

The docile primitives D_p are linear combinations of monomials that are balanced in x, y and such that $\deg_x \leq \deg_\epsilon$ and $\deg_a \leq \deg_\epsilon$ for each monomial. The D_p forms a sub-Hopf algebra of QU. This is because we may assume elements of D_p are in standard yax order. Commuting generators to bring the product again in yax form does not change the defining properties of D_p. The same goes for coproduct and antipode. All the operations produce $A = e^{-\hbar\epsilon a}$, never a without ϵ.

3.2 Computations with exponentials

The generating function $H = e^{\eta y} e^{\alpha a} e^{\xi x}$ is central to our computations. Any element of QU can be encoded by a 3-variable commutative power series f via $\mathbb{O}f = f(\partial_\eta, \partial_\alpha, \partial_\xi)H|_{\eta,\alpha,\xi=0}$. For example $y^2x + 2a = (\partial_\eta^2 \partial_\xi + 2\partial_\alpha)H|_{\eta,\alpha,\xi=0}$. This reduces all non-commutative computations to those involving the exponentials in H only.

$xa = (a-1)x$ so for any series $f(a)$ we have $x^n f(a) = f(a-n)x^n$, so $e^{\xi x}\epsilon^{\alpha a} = \sum_n \frac{\xi^n}{n!}\epsilon^{\alpha(a-n)}x^n = e^{\alpha a}e^{e^{-\alpha}\xi x}$. Similarly $e^{\alpha a}e^{\eta y} = e^{e^{-\alpha}\eta y}e^{\alpha a}$.

To swap exponentials in x, y we compute $F = e^{-\eta y}e^{\xi x}e^{\eta y}e^{-\xi x}$. In other words, find F such that $e^{\xi x}e^{\eta y} = e^{\eta y}F\eta^{\xi x}$. Set $G = e^{\xi x}ye^{-\xi x}$ so $F = e^{-\eta y}e^{\eta H}$. Notice that F solves the differential equation

$$\partial_\eta F = -yF + FG \qquad F|_{\eta=0} = 1$$

Notice that G itself also satisfies a differential equation: $\partial_\xi G = xG + Gx$ with $G(0) = y$.

Writing $F = \sum_{\ell,i,j,k} f_{i,j,k,\ell}(\eta) y^i a^j x^k \epsilon^\ell$ we see that our differential equation comes down to linear differential equations expressing $f'_{i,j,k,\ell}$ as a combination of $f_{i',j',k',\ell'}$ such that $i' + j' + k' + \ell' \leq i + j + k + \ell$.

When $\epsilon^\kappa = 0$ for some finite κ we see that F must be in the finite dimensional vector space spanned by elements $F = \sum_{\ell,i,j,k} f_{i,j,k,\ell}(\eta) y^i a^j x^k \epsilon^\ell$.