Solvable Approximations of the Quantum sl_2 Portfolio

With Roland van der Veen

ωεβ:=http://drorbn.net/mm18/

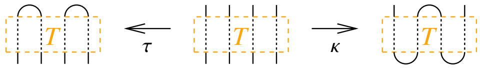
Our Main Theorem (loosely stated). Everything that matters in the quantum sl_2 portfolio can be continuously expressed in terms of docile perturbed Gaussians using solvable approximations. ○

Our Main Points.

- What's the "quantum sl_2 portfolio"?
- What in it "matters" and why? (the most important question)
- What's "solvable approximation"? What's "continuously"?
- What are "docile perturbed Gaussians"?
- Why do they matter? (2^{nd} most important)
- How proven? (docile)
- How implemented? (sacred; the work of unsung heroes)
- Some context and background.
- What's next?

The quantum sl_2 Portfolio includes a classical universal enveloping algebra CU, its quantization QU, their tensor powers $CU^{\otimes S}$ and $QU^{\otimes S}$ with the "tensor operations" ⊗, their products m_k^{ij}, coproducts Δ_{jk}^i and antipodes S_i, their Cartan automophisms $C\theta\colon CU \to CU$ and $Q\theta\colon QU \to QU$, the "dequantizators" $\mathbb{AD}\colon QU \to CU$ and $\mathbb{SD}\colon QU \to CU$, and most importantly, the R-matrix R and the Drinfel'd element s. All this in any PBW basis, and change of basis maps are included.

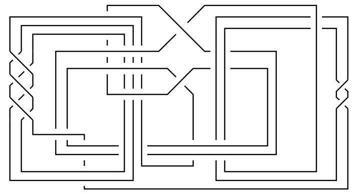
$$R, s \in \{QU^{\otimes S}\} \xrightarrow{\mathbb{AD}, \mathbb{SD}} \{CU^{\otimes S}\}$$



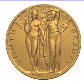
Gompf, Scharlemann, Thompson [GST]

$U \in \mathcal{T}_n$, $1 \in \mathcal{A}_n$, $\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n}$, with $\mathcal{R} := \kappa(\tau^{-1}(1))$

ribbon $K \in \mathcal{T}_1$ $\quad z(K) \in \mathcal{R} \subseteq \mathcal{A}_1$



Faster is better, leaner is meaner!



The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has

$$\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A \end{array}\right\} \text{ with } R_S := \mathbb{Z}(\{t_a\colon a \in S\}):$$

$$\left({}_a\nearrow_b, {}_b\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm1} \\ b & 0 & t_a^{\pm1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a, t_b \to t_c]{m_c^{ab}} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}\right)$$

(Roland: "add to A the product of column b and row a, divide by $(1 - A_{ab})$, delete column b and row a".)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

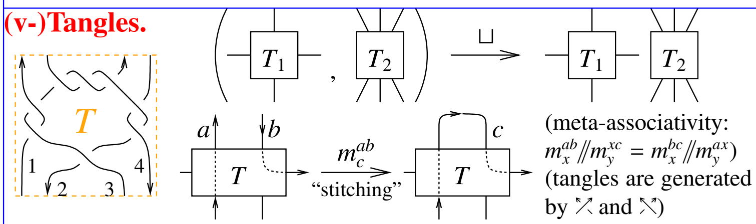
$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow[\substack{\mu := T_a - 1 \\ \nu := \alpha - \sigma_a \\ T_a \mapsto T_b T_c}]{q\Delta_{bc}^a} \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b - 1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu \\ c & (T_c - 1)\nu/\mu & (\alpha - \sigma_a T_a - \nu T_c)/\mu & (T_c - 1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array}\right)$$

$$\Big\downarrow dS^a,\ T_a \to T_a^{-1}$$

$$\left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha \end{array}\right)$$

Where σ assigns to every $a \in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left({}_a\nearrow_b, {}_b\nwarrow_a\right) = (a \to 1, b \to t_a^{\pm1})$, $\sigma(T_1 \sqcup T_2) = \sigma(T_1) \sqcup \sigma(T_2)$, and $\sigma /\!/ m_c^{ab} = (\sigma \setminus \{a, b\}) \cup (c \to \sigma_a\sigma_b)|_{t_a, t_b \to t_c}$.

(v-)Tangles.

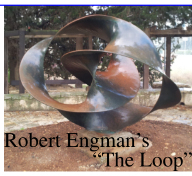


(meta-associativity: $m_x^{ab} /\!/ m_y^{xc} = m_x^{bc} /\!/ m_y^{ax}$) (tangles are generated by ↗ and ↖)

Strand doubling: $a \xrightarrow{\Delta_{bc}^a}$ b, c $\qquad$ Strand reversal: $a \xrightarrow{S_a} a$

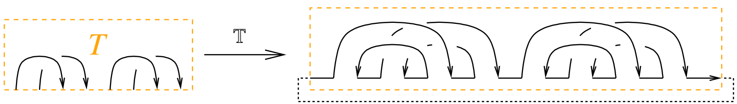
Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.



Robert Engman's "The Loop"

Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via



Implementation key idea: ωεβ/AlexDemo

$$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \textstyle\sum \alpha_{ab} t_a h_b)$$

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1*ω2, λ1+λ2];
m_{a_ b_ →c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  ( α β θ )   ( ∂_{t_a,h_a}λ  ∂_{t_a,h_b}λ  ∂_{t_a}λ )
  ( γ δ ϵ ) = ( ∂_{t_b,h_a}λ  ∂_{t_b,h_b}λ  ∂_{t_b}λ ) /. (t|h)_{a|b} → 0;
  ( ϕ ψ Ξ )   ( ∂_{h_a}λ      ∂_{h_b}λ      λ        )
  Γ[(μ = 1 - β) ω, {t_c, 1}.( γ+αδ/μ  ϵ+δθ/μ ).{h_c, 1}]
                             ( ϕ+αψ/μ  Ξ+ψθ/μ )
    /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_ b_} := Γ[1, {t_a, t_b}.( 1  1-T_a ).{h_a, h_b}];
                               ( 0  T_a   )
Rm_{a_ b_} := Rp_{ab} /. T_a → 1/T_a;

ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_{a_} :> a, ∞];
   M = Outer[Factor[∂_{h_{#1} t_{#2}} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];
```

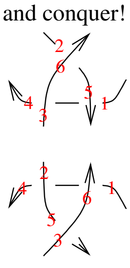
Meta-Associativity **Runs.**

```
ζ = Γ[ω, {t1, t2, t3, tS}.( α11 α12 α13 θ1 ).{h1, h2, h3, hS}];
                          ( α21 α22 α23 θ2 )
                          ( α31 α32 α33 θ3 )
                          ( ϕ1  ϕ2  ϕ3  Ξ  )
(ζ // m_{12→1} // m_{13→1}) == (ζ // m_{23→2} // m_{12→1})
True
```

R3 . . . divide and conquer!

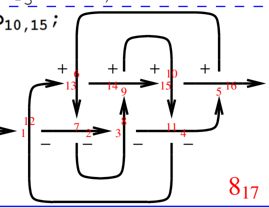
```
{Rm51 Rm62 Rp34 // m_{14→1} // m_{25→2} // m_{36→3},
 Rp61 Rm24 Rm35 // m_{14→1} // m_{25→2} // m_{36→3}}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}\right\}$$



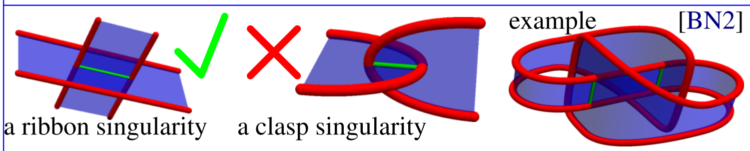

```
z = Rm_{12,1} Rm_{27} Rm_{83} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15};
Do[z = z // m_{1k→1}, {k, 2, 16}];
z
```

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1 \\ t_1 & 1 \end{pmatrix}$$



8_{17}

example [BN2]

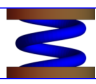


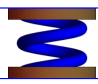
a ribbon singularity $\quad$ a clasp singularity

A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)





"God created the knots, all else in topology is the work of mortals."

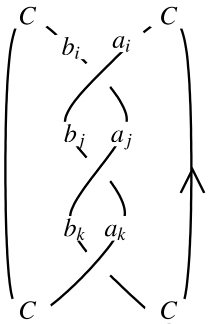
Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R=\sum a_i\otimes b_i\in U\otimes U \quad\text{and}\quad C\in U,$$

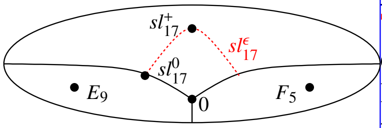
form $$Z=\sum_{i,j,k} Ca_ib_ja_kC^2b_ia_jb_kC.$$



Problem. Extract information from Z.

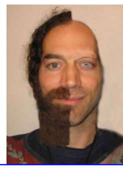
The Dogma. Use representation theory. In principle finite, but *slow*.

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2}\otimes V$ is on the right. We care about $sl^k_{17}:=sl^\epsilon_{17}/(\epsilon^{k+1}=0)$.

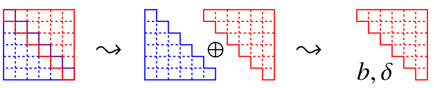


Solvable Approximation. A quantized universal enveloping algebra (aka "quantum group") is an ∞-dimensional inverse limit.

untrimmed	→	halfway-trimmed	→	almost fully-trimmed
↕		↕		↕
too hard		"solvable approximation" (a parameter is hidden)		"finite-type"



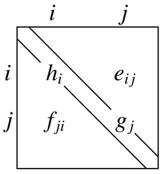
Recomposing gl_n. Half is enough! $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(◹,b,\delta)$:



$b(◹)=b\colon ◹\otimes◹\to◹$

$b(◺)\rightsquigarrow\delta\colon ◹\to◹\otimes◹$

Now define $gl^\epsilon_n:=\mathcal{D}(◹,b,\epsilon\delta)$. Schematically, this is $[◹,◹]=◹$, $[◺,◺]=\epsilon◺$, and $[◹,◺]=◺+\epsilon◹$. In detail, it is



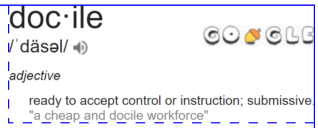
$$[e_{ij},e_{kl}]=\delta_{jk}e_{il}-\delta_{li}e_{kj} \qquad [f_{ij},f_{kl}]=\epsilon\delta_{jk}f_{il}-\epsilon\delta_{li}f_{kj}$$
$$[e_{ij},f_{kl}]=\delta_{jk}(\epsilon\delta_{j<k}e_{il}+\delta_{il}(h_i+\epsilon g_i)/2+\delta_{i>l}f_{il})$$
$$\qquad -\delta_{li}(\epsilon\delta_{k<j}e_{kj}+\delta_{kj}(h_j+\epsilon g_j)/2+\delta_{k>j}f_{kj})$$
$$[g_i,e_{jk}]=(\delta_{ij}-\delta_{ik})e_{jk} \qquad [h_i,e_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})e_{jk}$$
$$[g_i,f_{jk}]=(\delta_{ij}-\delta_{ik})f_{jk} \qquad [h_i,f_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})f_{jk}$$

Solvable Approximation (2). At $\epsilon=1$ and modulo $h=g$, the above is just gl_n. By rescaling at $\epsilon\neq 0$, gl^ϵ_n is independent of ϵ. We let gl^k_n be gl^ϵ_n regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1}=0$. It is the "$k$-smidgen solvable approximation" of gl_n!

Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2:=[\mathfrak{g},\mathfrak{g}]$, $\mathfrak{g}_3:=[\mathfrak{g}_2,\mathfrak{g}_2]$, ..., $\mathfrak{g}_d=0$. Equivalently, if it is a subalgebra of some large-size ◹ algebra.

Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

Definition. A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form

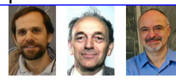


$$\mathbb{e}^{q^{ij}z_iz_j}P=\mathbb{e}^{q^{ij}z_iz_j}\left(\sum_{k\geq 0}\epsilon^kP_k\right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k\leq 4k$.

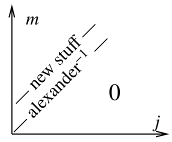
Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$.

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing



$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar}=\sum_{j,m\geq 0}a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.

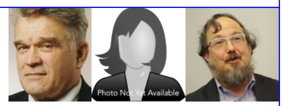


"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q)=\frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$

Prior art. Some amazing computations by Rozansky and Overbay in [Ro2, Ro3] and in [Ov].

Faddeev's Formula (In as much as we can tell, first appeared w/o proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q:=\frac{q^n-1}{q-1}$, with $[n]_q!:=[1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}^x_q:=\sum_{n\geq 0}\frac{x^n}{[n]_q!}$, we have



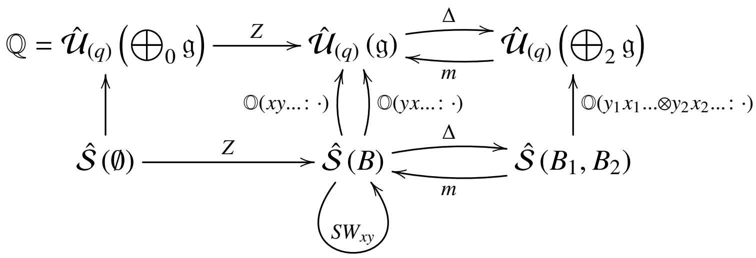
$$\log\mathbb{e}^x_q=\sum_{k\geq 1}\frac{(1-q)^kx^k}{k(1-q^k)}=x+\frac{(1-q)^2x^2}{2(1-q^2)}+\ldots.$$

Proof. We have that $\mathbb{e}^x_q=\frac{\mathbb{e}^{qx}_q-\mathbb{e}^x_q}{qx-x}$ ("the q-derivative of $\mathbb{e}^x_q$ is itself"), and hence $\mathbb{e}^{qx}_q=(1+(1-q)x)\mathbb{e}^x_q$, and

$$\log\mathbb{e}^{qx}_q=\log(1+(1-q)x)+\log\mathbb{e}^x_q.$$

Writing $\log\mathbb{e}^x_q=\sum_{k\geq 1}a_kx^k$ and comparing powers of x, we get $q^ka_k=-(1-q)^k/k+a_k$, or $a_k=\frac{(1-q)^k}{k(1-q^k)}$. □

$\mathcal{GDO}$-Categories. Given $\mathfrak{g}$ with basis $B=\{x,y,\ldots\}$, consider the following diagram:



Hence Z, SW_{xy}, m, Δ, (and likewise S and θ) are morphisms in the *completion* of the monoidal category $\mathcal{F}$ whose objects are finite sets B and whose morphisms are $\mathrm{mor}_{\mathcal{F}}(B,B'):=\mathrm{Hom}_{\mathbb{Q}}(\mathcal{S}(B)\to\mathcal{S}(B'))=\mathcal{S}(B^*,B')$ (by convention, $x^*=\xi$, $y^*=\eta$, etc.). Ergo we need to *consolidate* (at least parts of) said completion.

Aside. "Consolidate" means "give a finite name to an infinite object, and figure out how to sufficiently manipulate such finite names". E.g., solving $f''=-f$ we encounter and set $\sum\frac{(-1)^kx^{2k}}{(2k)!}\rightsquigarrow\cos x$, $\sum\frac{(-1)^kx^{2k+1}}{(2k+1)!}\rightsquigarrow\sin x$, and then $\cos^2x+\sin^2x=1$ and $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

The Composition Law. If

$$\mathcal{S}(B_0)\xrightarrow[{}^tf\in\mathbb{Q}[\![\zeta_{0i},z_{1j}]\!]]{f}\mathcal{S}(B_1)\xrightarrow[{}^tg\in\mathbb{Q}[\![\zeta_{1j},z_{2k}]\!]]{g}\mathcal{S}(B_2)$$

then ${}^t(f/\!/g)={}^t(g\circ f)=\left(g|_{\zeta_{1j}\to\partial_{z_{1j}}}f\right)_{z_{1j}=0}$.

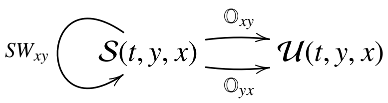
Examples.

1. The 1-variable identity map $I\colon\mathcal{S}(z)\to\mathcal{S}(z)$ is given by ${}^tI_1=\mathbb{e}^{z\zeta}$ and the n-variable one by ${}^tI_n=\mathbb{e}^{z_1\zeta_1+\cdots+z_n\zeta_n}$.

2. The "$z_i \to z_j$ variable rename map $\sigma^i_j\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j)$ becomes ${}^t\sigma^i_j = \mathbb{e}^{z_j\zeta_i}$, and it's easy to rename several variables simultaneously.
3. The "archetypal multiplication map $m^{ij}_k\colon \mathcal{S}(z_i, z_j) \to \mathcal{S}(z_k)$" has ${}^t m = \mathbb{e}^{z_k(\zeta_i+\zeta_j)}$.
4. The "archetypal coproduct $\Delta^i_{jk}\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j, z_k)$", given by $z_i \to z_j + z_k$ or $\Delta z = z\otimes 1 + 1\otimes z$, has ${}^t\Delta = \mathbb{e}^{(z_j+z_k)\zeta_i}$.
5. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1 x_2} \in \mathcal{U}_q \otimes \mathcal{U}_q$. The "baby R-matrix" is ${}^t R = \mathbb{e}^{\hbar yx} \in \mathcal{S}(y, x)$.

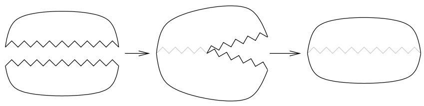
Proposition. If $F\colon \mathcal{S}(B) \to \mathcal{S}(B')$ is linear and "continuous", then ${}^t F = \exp\left(\sum_{z_i\in B} \zeta_i z_i\right) /\!\!/ F$.

The Heisenberg Example. The "Weyl form of the canonical commutation relations" states that if $[y, x] = t$ and t is central, then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. Thus with



we have ${}^t SW_{xy} = \mathbb{e}^{\tau t+\eta y+\xi x-\eta\xi t}$.

The Zipping Issue (between unbound and bound lies half-zipped).



Zipping. If $P(\zeta^j, z_i)$ is a polynomial, or whenever otherwise convergent, set

$$\left\langle P(\zeta^j, z_i)\right\rangle_{(\zeta^j)} = P\left(\partial_{z_j}, z_i\right)\Big|_{z_i=0}.$$

(E.g., if $P = \sum a_{nm}\zeta^n z^m$ then $\langle P\rangle_\zeta = \sum n! a_{nn}$).

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i, \zeta^j)\mathbb{e}^{\eta^i z_i + y_j\zeta^j}\right\rangle_{(\zeta^j)} = \left\langle P(z_i + y_i, \zeta^j)\mathbb{e}^{\eta^i(z_i+y_i)}\right\rangle_{(\zeta^j)},$$

(proof: replace $y_j \to \hbar y_j$ and test at $\hbar = 0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i, \zeta^j)\mathbb{e}^{c+\eta^i z_i + y_j\zeta^j + q^i_j z_i\zeta^j}\right\rangle_{(\zeta^j)}$$
$$= \det(\tilde q)\left\langle P(\tilde q^k_i(z_k + y_k), \zeta^j)\mathbb{e}^{c+\eta^i\tilde q^k_i(z_k+y_k)}\right\rangle_{(\zeta^j)}$$

where $\tilde q$ is the inverse matrix of $1 - q$: $(\delta^i_j - q^i_j)\tilde q^j_k = \delta^i_k$ (proof: replace $q^i_j \to \hbar q^i_j$ and test at $\hbar = 0$ and at $\partial_\hbar$).

Implementation. ωεβ/ZipBindDemo

```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
{z*, x*, y*} = {ζ, ξ, η}; {ζ*, ξ*, η*} = {z, x, y};
(u_i_)* := (u*)_i;
```

```
Zip_{}[P_] := P;
Zip_{ζ_,ζs___}[P_] :=
  (Expand[P // Zip_{ζs}] /. f_. ζ^d_. :> ∂_{ζ*,d} f) /. ζ* → 0
```

```
Zip_{ζ}[(a ζ^6 + ζ + 3) (z^5 e^z + 7 z) + 99 b]
```

```
7 + 720 a + 99 b
```

```
Zip_{ξ,η}[ξ^3 η^3 e^(a x + b y + c x y)]
```

```
a^3 b^3 + 9 a^2 b^2 c + 18 a b c^2 + 6 c^3
```

```
(* 𝔼[Q,P] means e^Q P *)
𝔼 /: Zip_{ζs_List}@𝔼[Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q1 = c + ηs.zs /. zrule;
   Q2 = Q1 /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[Q2, Det[qt] e^-Q2 Zip_{ζs}[e^Q1 (P /. zrule)]] ];
```

```
Eh = 𝔼[h Sum[Sum[a_{10 i+j} x_i ξ_j, {j,1,3}], {i,1,3}], Sum[f_i[x_1, x_2, x_3] ξ_i, {i,1,3}]];
E1 = Eh /. h → 1
```

```
𝔼[a_11 x_1 ξ_1 + a_21 x_2 ξ_1 + a_31 x_3 ξ_1 + a_12 x_1 ξ_2 +
  a_22 x_2 ξ_2 + a_32 x_3 ξ_2 + a_13 x_1 ξ_3 + a_23 x_2 ξ_3 + a_33 x_3 ξ_3,
 ξ_1 f_1[x_1, x_2, x_3] + ξ_2 f_2[x_1, x_2, x_3] + ξ_3 f_3[x_1, x_2, x_3]]
```

```
Short[lhs = Zip_{ξ1,ξ2}@E1, 5]
```

```
𝔼[((a_13 ((-1 + a_22) a_31 - a_21 a_32) + a_12 (-a_23 a_31 + a_21 a_33) +
      (-1 + a_11) (a_23 a_32 - (-1 + a_22) a_33)) x_3 ξ_3) /
   (-1 + a_12 a_21 - a_11 (-1 + a_22) + a_22),
  (<<17>> + a_21 <<1>>) / (-1 + a_12 a_21 - a_11 (-1 + a_22) + a_22)^2]
```

```
lhs == Zip_{ξ1}@Zip_{ξ2}@E1 == Zip_{ξ2}@Zip_{ξ1}@E1
```

```
True
```

```
Short[
 lhs = Normal[Eh /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 3}]] //
   Zip_{ξ1,ξ2}, 5]
```

```
h a_13 ξ_3 f_1[0, 0, x_3] + 2 h^2 a_11 a_13 ξ_3 f_1[0, 0, x_3] +
 3 h^3 a_11^2 a_13 ξ_3 f_1[0, 0, x_3] + 2 h^3 a_12 a_13 a_21 ξ_3 f_1[0, 0, x_3] +
 h^2 a_13 a_22 ξ_3 f_1[0, 0, x_3] + <<337>> +
 1/6 h^3 a_31^3 x_3^3 ξ_3 f_3^(3,0,0)[0, 0, x_3] + 1/2 h^3 a_31^2 a_32 x_3^3 f_1^(3,1,0)[0, 0, x_3] +
 1/6 h^3 a_31^3 x_3^3 f_2^(3,1,0)[0, 0, x_3] + 1/6 h^3 a_31^3 x_3^3 f_1^(4,0,0)[0, 0, x_3]
```

```
rhs =
  Normal[Zip_{ξ1,ξ2}@Eh /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 3}]];
Simplify[lhs == rhs]
```

```
True
```

```
𝔼 /: 𝔼[Q1_, P1_] 𝔼[Q2_, P2_] := 𝔼[Q1 + Q2, P1 * P2];
Bind_{ζs_List}[L_𝔼, R_𝔼] := Module[{n, hideζs, hidezs},
   hideζs = Table[ζs⟦i⟧ → ζ_{n@i}, {i, Length@ζs}];
   hidezs = Table[ζs⟦i⟧* → z_{n@i}, {i, Length@ζs}];
   Zip_{ζs/.hideζs}[(L /. hidezs) (R /. hideζs)] ];
```

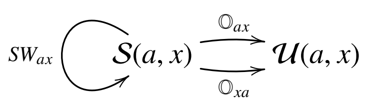
```
Bind_{ξ2}[𝔼[ξ (x_1 + x_2), 1], 𝔼[ξ_2 (x_2 + x_3), 1]]
```

```
𝔼[ξ (x_1 + x_2 + x_3), 1]
```

```
Bind_{ξ2}[𝔼[(ξ_2 + ξ_3) x_2, 1], 𝔼[(ξ_1 + ξ_2) x, 1]]
```

```
𝔼[x (ξ_1 + ξ_2 + ξ_3), 1]
```

The 2D Lie Algebra. Clever people know* that if $[a, x] = \gamma x$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$. Ergo with



we have ${}^t SW_{ax} = \mathbb{e}^{\alpha a+\mathbb{e}^{-\gamma\alpha}\xi x}$.

∗ Indeed $xa = (a-\gamma)x$ thus $xa^n = (a-\gamma)^n x$ thus $x\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha(a-\gamma)}x = \mathbb{e}^{-\gamma\alpha}\mathbb{e}^{\alpha a}x$ thus $x^n\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}(\mathbb{e}^{-\gamma\alpha})^n x^n$ thus $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$.

The Real Thing. In $QU/(\epsilon^2 = 0)$ over $\mathbb{Q}[\![\hbar]\!]$ using the yax order, $T = \mathbb{e}^{\hbar t}$, $\bar T = T^{-1}$, $\mathcal{A} = \mathbb{e}^{\gamma\alpha}$, and $\bar{\mathcal{A}} = \mathcal{A}^{-1}$, we have

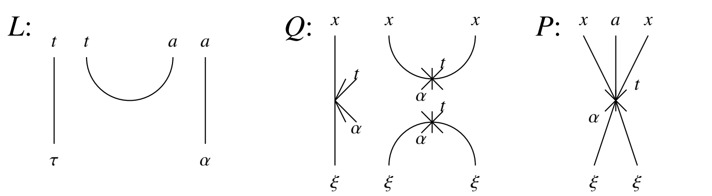
$${}^t R_{ij} = \mathbb{e}^{\hbar(y_i x_j - t_i a_j/\gamma)}\left(1 + \epsilon\hbar\left(a_i a_j/\gamma - \gamma\hbar^2 y_i^2 x_j^2/4\right)\right)$$

in $\mathcal{S}(B_i, B_j)$, and in $\mathcal{S}(B_1^*, B_2^*, B)$ we have

$${}^t m = \mathbb{e}^{(\alpha_1+\alpha_2)a+\eta_2\xi_1(1-T)/\hbar+(\xi_1\bar{\mathcal{A}}_2+\xi_2)x+(\eta_1+\eta_2\bar{\mathcal{A}}_1)y}(1+\epsilon\lambda_m),$$

where $\lambda_m = 2a\eta_2\xi_1 T + \frac14\gamma\eta_2^2\xi_1^2\left(3T^2-4T+1\right)/\hbar - \frac12\gamma\eta_2\xi_1^2(3T-1)x\bar{\mathcal{A}}_2 - \frac12\gamma\eta_2^2\xi_1(3T-1)y\bar{\mathcal{A}}_1 + \gamma\eta_2\xi_1 xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$. Similar formulas delight us for ${}^t\Delta$ and ${}^t S$.

A generic morphism.



Implementation.


```
QZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   simp /@ 𝔼[L, Q2, Det[qt] e^{-Q2} Zip_ζs[e^{Q1} (P /. zrule)]] ];
QZip_{ζs_List} := QZip_{ζs,CF};
```

```
LZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Q2 = (Q1 = Q /. T2t /. zrule) /. Alternatives @@ zs → 0;
   simp /@
    𝔼[L2, Q2, Det[lt] e^{-L2-Q2}
      Zip_ζs[e^{L1+Q1} (P /. T2t /. zrule)]] //. t2T ];
LZip_{ζs_List} := LZip_{ζs,CF};
```

```
Bind_{}[L_, R_] := L R;
Bind_{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : T | t | a | x | y)_i → v_{n@i}, {i, {is}}],
     R /. Table[(v : τ | α | ξ | η)_i → v_{n@i}, {i, {is}}]
    ] // LZip_{Flatten@Table[{τ_{n@i},a_{n@i}},{i,{is}}]} //
   QZip_{Flatten@Table[{ξ_{n@i},y_{n@i}},{i,{is}}]} ];
B_{l_List} := Bind_l; B_{is___} := Bind_{{is}};
Bind[ℰ_𝔼] := ℰ;
Bind[Ls__, ζs_List, R_] := Bind_ζs[Bind[Ls], R];
```

A Partial To Do List.

- Complete all "docility" arguments by identifying a "contained" docile substructure.
- Understand denominators and get rid of them.
- See if much can be gained by including P in the exponential: $\mathbb{e}^{L+Q}P \rightsquigarrow \mathbb{e}^{L+Q+P}$?
- Clean the program and make it efficient.
- Run it for all small knots and links, at $k = 2, 3$.
- Understand the centre and figure out how to read the output.
- Execute the Drinfel'd double procedure at $\mathbb{E}$-level (and thus get rid of `DeclareAlgebra` and all that is around it!).
- Extend to sl_3 and beyond.
- Do everything with `Zip` and `Bind` as the fundamentals, without ever referring back to (quantized) Lie algebras.
- Prove a genus bound and a Seifert formula.
- Obtain "Gauss-Gassner formulas" (ωεβ/NCSU).
- Relate with Melvin-Morton-Rozansky and with Rozansky-Overbay.
- Understand the braid group representations that arise.
- Find a topological interpretation. The Garoufalidis-Rozansky "loop expansion" [GR]?
- Figure out the action of the Cartan automorphism.
- Disprove the ribbon-slice conjecture!
- Figure out the action of the Weyl group.
- Do everything at the "arrow diagram" level of finite-type invariants of (rotational) virtual tangles.
- What else can you do with the "solvable approximations"?
- And with the "Gaussian zip and bind" technology?

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The Complete Implementation.

ωεβ/SL2Portfolio

An even fuller implementation is at ωεβ/FullImp.

Initialization / Utilities

```
$p = 2; $k = 1; $U = QU; $E := {$k, $p};
$trim := {ħ^{p_·} /; p > $p → 0, ϵ^{k_·} /; k > $k → 0};
q_ħ = e^{γ ϵ ħ};
T2t = {T_{i_}^{p_·} → e^{p ħ t_i}, T^{p_·} → e^{p ħ t}};
t2T = {e^{c_· t_{i_}+b_·} :→ T_i^{c/ħ} e^b, e^{c_· t+b_·} :→ T^{c/ħ} e^b, e^{ℰ_} :→ e^{Expand@ℰ}};
SetAttributes[SS, HoldAll];
SS[ℰ_, op_] := Collect[
   Normal@Series[If[$p > 0, ℰ, ℰ /. T2t], {ħ, 0, $p}],
   ħ, op];
SS[ℰ_] := SS[ℰ, Together];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
c_Integer_{k_Integer} := c + O[ϵ]^{k+1};
```

```
CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@
    Together[Expand[ℰ] //. e^{x_} e^{y_} :→ e^{x+y} /. e^{x_} :→ e^{CF[x]}];
```

```
Unprotect[SeriesData];
SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
SeriesData /: Expand[sd_SeriesData] :=
  MapAt[Expand, sd, 3];
SeriesData /: Simplify[sd_SeriesData] :=
  MapAt[Simplify, sd, 3];
SeriesData /: Together[sd_SeriesData] :=
  MapAt[Together, sd, 3];
SeriesData /: Collect[sd_SeriesData, specs__] :=
  MapAt[Collect[#, specs] &, sd, 3];
Protect[SeriesData];
```

DeclareAlgebra

```
Unprotect[NonCommutativeMultiply];
Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ** y) ** z;
0 ** _ = _ ** 0 = 0;
(x_Plus) ** y_ := (# ** y) & /@ x;
x_ ** (y_Plus) := (x ** #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ** y - y ** x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
DeclareAlgebra[U_Symbol, opts__Rule] :=
 Module[{gp, sr, g, cp, M, CE, k = 0,
   gs = Generators /. {opts},
   cs = Centrals /. {opts} /. Centrals → {} },
  (#_U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
  sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}];
  (* sorting → *)
  cp = Alternatives @@ cs; (* cents *)
  SetAttributes[M, HoldRest]; M[0, _] = 0;
  M[a_, x_] := a x;
  CE[ℰ_] := Collect[ℰ, _U, Expand] /. $trim;
  U_i_[ℰ_] := ℰ /. {t : cp ⧴ t_i, u_U ⧴ (#_i &) /@ u};
  U_i_[NCM[]] = U@{} = 1_U = U[];
  B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
  B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ** (c_. 1_U) := CE[c x]; (c_. 1_U) ** x_ := CE[c x];
  (a_. U[xx___, x_]) ** (b_. U[y_, yy___]) :=
   If[OrderedQ[{x, y} /. sr],
    CE@M[a b /. $trim, U[xx, x, y, yy]],
    U@xx **
     CE@M[a b /. $trim, U@y ** U@x + B[U@x, U@y, $E]] **
     U@yy ];
  U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] :=
   CE[c U@Table[l, {n}] ** U@{r}];
  U@{c_. * l : gp, r___} := CE[c U[l] ** U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
  U@{l_, r___} := U@{Expand[l], r};
  U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
  𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, l_List ⧴ l_null, {1}];
    vs = Join @@ (First /@ sp);
    us = Join @@ (sp /. l_s_ ⧴ (l /. x_i_ ⧴ x_s));
    CE[Total[
       CoefficientRules[poly, vs] /. (p_ → c_) ⧴ c U@(us^p)
      ]] /. x_null ⧴ x];
  𝕆_U[specs___, 𝔼[L_, Q_, P_]] :=
   𝕆_U[specs, SS@Normal[P e^(L+Q)]];
  σ_rs___[c_. * u_U] :=
   (c /. (t : cp)_j_ ⧴ t_(j/.{rs})) U[List @@ (u /. v_j_ ⧴ v_(j/.{rs}))];
  m_j_→k_[c_. * u_U] :=
   CE[((c /. (t : cp)_j → t_k) DeleteCases[u, _j|k]) **
     U @@ Cases[u, w_j ⧴ w_k] ** U @@ Cases[u, _k]];
  U /: c_. * u_U * v_U := CE[c u ** v];
  S_i_[c_. * u_U] :=
   CE[((c /. S_i[U, Centrals]) DeleteCases[u, _i]) **
     U_i[NCM @@ Reverse@Cases[u, x_i ⧴ S@U@x]]];
  Δ_i_→j_,k_[c_. * u_U] :=
   CE[((c /. Δ_i→j,k[U, Centrals]) DeleteCases[u, _i]) **
     (NCM @@ Cases[u, x_i ⧴ σ_1→j,2→k@Δ@U@x] /.
       NCM[] → U[])]; ]
```

DeclareMorphism

```
DeclareMorphism[m_, U_ → V_, ongs_List, oncs_List: {}] := (
  Replace[ongs, {(g_ → img_) ⧴ (m[U[g]] = img),
     (g_ ⧴ img_) ⧴ (m[U[g]] := img /. $trim)}, {1}];
  m[1_U] = 1_V;
  m[U[g_i_]] := V_i[m[U@g]];
  m[U[vs__]] := NCM @@ (m /@ U /@ {vs});
  m[ℰ_] := Simp[ℰ /. oncs /. u_U ⧴ m[u]] /. $trim; )
```

Meta-Operations

```
σ_rs___[ℰ_Plus] := σ_rs /@ ℰ;
m_j_→j_ = Identity; m_j_→k_[0] = 0;
m_j_→k_[ℰ_Plus] := Simp[m_j→k /@ ℰ];
m_is___,i_,j_→k_[ℰ_] := m_j→k@m_is,i→j@ℰ;
S_i_[ℰ_Plus] := Simp[S_i /@ ℰ];
Δ_is___[ℰ_Plus] := Simp[Δ_is /@ ℰ];
```

Implementing CU = $\mathcal{U}(sl_2^{\gamma\epsilon})$

```
DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[a_CU, y_CU] = -γ y_CU; B[x_CU, a_CU] = -γ x_CU;
B[x_CU, y_CU] = 2 ϵ a_CU - t 1_CU;
(S@y_CU = -y_CU; S@a_CU = -a_CU; S@x_CU = -x_CU;)
S_i_[CU, Centrals] = {t_i → -t_i};
Δ@y_CU = CU@y_1 + CU@y_2; Δ@a_CU = CU@a_1 + CU@a_2;
Δ@x_CU = CU@x_1 + CU@x_2;
Δ_i_→j_,k_[CU, Centrals] = {t_i → t_j + t_k};
```

Implementing QU = $\mathcal{U}_q(sl_2^{\gamma\epsilon})$

```
DeclareAlgebra[QU, Generators → {y, a, x},
  Centrals → {t, T}];
B[a_QU, y_QU] = -γ y_QU; B[x_QU, a_QU] = -γ QU@x;
B[x_QU, y_QU] := SS[q_ħ - 1] QU@{y, x} +
   𝕆_QU[{a}, SS[(1 - T e^(-2 ϵ a ħ)) / ħ]];
(S@y_QU := 𝕆_QU[{a, y}, SS[-T^-1 e^(ħ ϵ a) y]]; S@a_QU = -a_QU;
 S@x_QU := 𝕆_QU[{a, x}, SS[-e^(ħ ϵ a) x]];)
S_i_[QU, Centrals] = {t_i → -t_i, T_i → T_i^-1};
Δ@y_QU := 𝕆_QU[{y_1, a_1}_1, {y_2}_2, SS[y_1 + T_1 e^(-ħ ϵ a_1) y_2]];
Δ@a_QU = QU@a_1 + QU@a_2;
Δ@x_QU := 𝕆_QU[{a_1, x_1}_1, {x_2}_2, SS[x_1 + e^(-ħ ϵ a_1) x_2]];
Δ_i_→j_,k_[QU, Centrals] = {t_i → t_j + t_k, T_i → T_j T_k};
```

The representation ρ

```
ρ@y_CU = ρ@y_QU = (0 0
                   ϵ 0); ρ@a_CU = ρ@a_QU = (γ 0
                                             0 0);
ρ@x_CU = (0 γ
          0 0); ρ@x_QU = (0 (1 - e^(-γ ϵ ħ)) / (ϵ ħ)
                          0 0);
ρ[e^ℰ_] := MatrixExp[ρ[ℰ]];
ρ[ℰ_] :=
 (ℰ /. T2t /. t → γ ϵ /.
   (U : CU | QU)[u___] ⧴ Fold[Dot, (1 0
                                   0 1), ρ /@ U /@ {u}])
```

tSW

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta(e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
SW_xy[U_, kk_] :=
  SW_xy[U, kk] = Block[{$U = U, $k = kk, $p = kk},
    Module[{G, F, fs, f, bs, e, b, es},
     G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].
        NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
     fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, l},
        {j, 0, l}, {k, 0, l}];
     F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k});
     es = Flatten[Table[Coefficient[e, b] == 0,
        {e, {F - 1_U /. η → 0, F ** G - y_U ** F - ∂_η F}},
        {b, bs}]];
     F = F /. DSolve[es, fs, η][[1]];
     𝔼[0,
       ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
       F + O_$k /. {e- → 1, U → Times}
      ] /. (v : η | ξ | t | T | y | a | x) → v₁
    ]];
tSW_{xy_,i_,j_→k_} :=
  SW_xy[$U, $k] /. {ξ₁ → ξ_i, η₁ → η_j, (v : t | T | y | a | x)₁ → v_k};
tSW_{xa,i_,j_→k_} := 𝔼[α_j a_k, e^{-γ α_j} ξ_i x_k, 1];
tSW_{ay,i_,j_→k_} := 𝔼[α_i a_k, e^{-γ α_i} η_j y_k, 1];
```

Exponentials as needed.

Task. Define $\mathrm{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi\,\mathbb{O}(P)}$ to ϵ^k in the algebra U_i, where ξ is a scalar, X is x_i or y_i, and P is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form. Should satisfy $U@\mathrm{Exp}_{U_i,k}[\xi, P] == \mathbb{S}_U[e^{\xi x},\ x \to \mathbb{O}(P)]$.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\xi\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)$, then $F(\xi = 0) = 1$ and we have:

$$\mathbb{O}\left(e^{\xi P_0}(P_0 F(\xi) + \partial_\xi F\right) = \mathbb{O}\left(\partial_\xi e^{\xi P_0} F(\xi)\right) =$$

$$\partial_\xi \mathbb{O}\left(e^{\xi P_0} F(\xi)\right) = \partial_\xi e^{\xi\,\mathbb{O}(P)} = e^{\xi\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\left(e^{\xi P_0} F(\xi)\right)\mathbb{O}(P)$$

This is an ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ.

```
(* Bug: The first line is valid only if O(e^P0) == e^O(P0). *)
(* Bug: ξ must be a symbol. *)
Exp_{U_i_,0}[ξ_, P_] := Module[{LQ = Normal@P /. ϵ → 0},
   𝔼[ξ LQ /. (x | y)_i → 0, ξ LQ /. (t | a)_i → 0, 1]];
Exp_{U_i_,k_}[ξ_, P_] := Block[{$U = U, $k = k},
  Module[{P0, φ, φs, F, j, rhs, at0, atξ},
   P0 = Normal@P /. ϵ → 0;
   φs = Flatten@Table[φ_{j1,j2,j3}[ξ], {j2, 0, k},
      {j1, 0, 2 k + 1 - j2}, {j3, 0, 2 k + 1 - j2 - j1}];
   F = Normal@Last@Exp_{U_i,k-1}[ξ, P] +
     ϵ^k φs.(φs /. φ_{js__}[ξ] :> Times @@ {y_i, a_i, x_i}^{js});
   rhs =
    Normal@
     Last@
      m_{i,j→i}[𝔼[ξ P0 /. (x | y)_i → 0, ξ P0 /. (t | a)_i → 0, F + O_k]
        m_{i→j}@𝔼[0, 0, P + O_k]];
   at0 = (# == 0) & /@
     Flatten@CoefficientList[F - 1 /. ξ → 0, {y_i, a_i, x_i}];
   atξ = (# == 0) & /@
     Flatten@CoefficientList[(∂_ξ F) + P0 F - rhs,
       {y_i, a_i, x_i}];
   𝔼[ξ P0 /. (x | y)_i → 0, ξ P0 /. (t | a)_i → 0, F + O_k] /.
    DSolve[And @@ (at0 ⋃ atξ), φs, ξ][[1]] ]]
```

Zip and Bind

```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] :=
  𝔼[L1 + L2, Q1 + Q2, P1 * P2];
{t*, y*, a*, x*, z*} = {τ, η, α, ξ, ζ};
{τ*, η*, α*, ξ*, ζ*} = {t, y, a, x, z};
(u_i_)* := (u*)_i;
Zip_{}[P_] := P;
Zip_{ζ_,ζs___}[P_] :=
 (Expand[P // Zip_{ζs}] /. f_. ζ^d_. :> ∂_{ζ*,d} f) /. ζ* → 0
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

```
QZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   simp /@ 𝔼[L, Q2, Det[qt] e^{-Q2} Zip_{ζs}[e^{Q1} (P /. zrule)]] ];
QZip_{ζs_List} := QZip_{ζs,CF};
```

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^{Q} as a single”P”. Here the z’s are t and α and the ζ’s are τ and a.

```
LZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Q2 = (Q1 = Q /. T2t /. zrule) /. Alternatives @@ zs → 0;
   simp /@
    𝔼[L2, Q2, Det[lt] e^{-L2-Q2}
      Zip_{ζs}[e^{L1+Q1} (P /. T2t /. zrule)] //. t2T ];
LZip_{ζs_List} := LZip_{ζs,CF};
Bind_{}[L_, R_] := L R;
Bind_{is__}[L_𝔼, R_𝔼] := Module[{n},
  Times[
    L /. Table[(v : T | t | a | x | y)_i → v_{n@i}, {i, {is}}],
    R /. Table[(v : τ | α | ξ | η)_i → v_{n@i}, {i, {is}}]
   ] // LZip_{Flatten@Table[{τ_{n@i}, a_{n@i}}, {i, {is}}]} //
   QZip_{Flatten@Table[{ξ_{n@i}, y_{n@i}}, {i, {is}}]} ];
B_{L_List} := Bind_L;  B_{is___} := Bind_{is};
Bind[ℰ_𝔼] := ℰ;
Bind[Ls__, ζs_List, R_] := Bind_{ζs}[Bind[Ls], R];
```

Tensorial Representations

```
tη = t1 = 𝔼[0, 0, 1 + O_$k];
tm_{i_,j_→k_} := Module[{tk},
   𝔼[(τ_i + τ_j) t_k + α_i a_k + α_j a_k, η_i y_k + ξ_j x_k, 1]
    (tSW_{xy,i,j→tk} /. {t_tk → t_k, T_tk → T_k, y_tk → e^{-γ α_i} y_k,
       a_tk → a_k, x_tk → e^{-γ α_j} x_k})];
m_{j_→k_}[ℰ_𝔼] := ℰ ~B_{j,k}~ tm_{j,k→k};
tm_{1,2→3}
```

$$\mathbb{E}\Big[a_3\,\alpha_1 + a_3\,\alpha_2 + t_3\,(\tau_1+\tau_2),$$
$$y_3\,\eta_1 + e^{-\gamma\alpha_1}\,y_3\,\eta_2 + e^{-\gamma\alpha_2}\,x_3\,\xi_1 + \frac{(1-T_3)\,\eta_2\,\xi_1}{\hbar} + x_3\,\xi_2,$$
$$1 + \frac{1}{4\hbar}\,\eta_2\,\xi_1\,\big(8\hbar\,a_3\,T_3 + 4\,e^{-\gamma\alpha_1-\gamma\alpha_2}\,\gamma\,\hbar^2\,x_3\,y_3 + 2\,e^{-\gamma\alpha_1}\,\gamma\,\hbar\,y_3\,\eta_2 - 6\,e^{-\gamma\alpha_1}\,\gamma\,\hbar\,T_3\,y_3\,\eta_2 + 2\,e^{-\gamma\alpha_2}\,\gamma\,\hbar\,x_3\,\xi_1 - 6\,e^{-\gamma\alpha_2}\,\gamma\,\hbar\,T_3\,x_3\,\xi_1 + \gamma\,\eta_2\,\xi_1 - 4\,\gamma\,T_3\,\eta_2\,\xi_1 + 3\,\gamma\,T_3^2\,\eta_2\,\xi_1\big)\,\epsilon + O[\epsilon]^2\Big]$$

```
S[U_, kk_] := S[U, kk] = Module[{OE},
   OE = m_{3,2,1→1}[Exp_{QU_1,$k}[η, S_1[QU[y_1]] /. QU → Times]
       Exp_{QU_2,$k}[α, S_2[QU[a_2]] /. QU → Times]
       Exp_{QU_3,$k}[ξ, S_3[QU[x_3]] /. QU → Times]];
   𝔼[-t_1 τ_1 + OE⟦1⟧, OE⟦2⟧, OE⟦3⟧] /.
     {η → η_1, α → α_1, ξ → ξ_1}];
tS_i_ := S[$U, $k] /. {(v : τ | η | α | ξ)_1 → v_i,
   (v : t | T | y | a | x)_1 → v_i};
```

```
tS_1
```

$$\mathbb{E}\Big[-a_1\,\alpha_1 - t_1\,\tau_1,$$
$$\frac{-e^{\gamma\alpha_1}\,\hbar\,y_1\,\eta_1 - e^{\gamma\alpha_1}\,\hbar\,T_1\,x_1\,\xi_1 + e^{\gamma\alpha_1}\,\eta_1\,\xi_1 - e^{\gamma\alpha_1}\,T_1\,\eta_1\,\xi_1}{\hbar\,T_1},\ 1 +$$
$$\frac{1}{4\hbar\,T_1^2}\,\big(4\,e^{\gamma\alpha_1}\,\gamma\,\hbar^2\,T_1\,y_1\,\eta_1 - 4\,e^{\gamma\alpha_1}\,\hbar^2\,a_1\,T_1\,y_1\,\eta_1 - 2\,e^{2\gamma\alpha_1}\,\gamma\,\hbar^2\,y_1^2\,\eta_1^2 - 4\,e^{\gamma\alpha_1}\,\hbar^2\,a_1\,T_1^2\,x_1\,\xi_1 - 4\,e^{\gamma\alpha_1}\,\gamma\,\hbar\,T_1\,\eta_1\,\xi_1 + 8\,e^{\gamma\alpha_1}\,\hbar\,a_1\,T_1\,\eta_1\,\xi_1 + 4\,e^{\gamma\alpha_1}\,\gamma\,\hbar\,T_1^2\,\eta_1\,\xi_1 - 4\,e^{2\gamma\alpha_1}\,\gamma\,\hbar^2\,T_1\,x_1\,y_1\,\eta_1\,\xi_1 + 6\,e^{2\gamma\alpha_1}\,\gamma\,\hbar\,y_1\,\eta_1^2\,\xi_1 - 2\,e^{2\gamma\alpha_1}\,\gamma\,\hbar\,T_1\,y_1\,\eta_1^2\,\xi_1 - 2\,e^{2\gamma\alpha_1}\,\gamma\,\hbar^2\,T_1^2\,x_1^2\,\xi_1^2 + 6\,e^{2\gamma\alpha_1}\,\gamma\,\hbar\,T_1\,x_1\,\eta_1\,\xi_1^2 - 2\,e^{2\gamma\alpha_1}\,\gamma\,\hbar\,T_1^2\,x_1\,\eta_1\,\xi_1^2 - 3\,e^{2\gamma\alpha_1}\,\gamma\,\eta_1^2\,\xi_1^2 + 4\,e^{2\gamma\alpha_1}\,\gamma\,T_1\,\eta_1^2\,\xi_1^2 - e^{2\gamma\alpha_1}\,\gamma\,T_1^2\,\eta_1^2\,\xi_1^2\big)\,\epsilon + O[\epsilon]^2\Big]$$

```
Δ[U_, kk_] := Δ[U, kk] = Module[{OE},
   OE = Block[{$k = kk, $p = kk + 1},
     m_{1,3,5→1}@
      m_{2,4,6→2}@Times[ (* Warning:
         wrong unless $p≥$k+1! *)
        ReplacePart[1 → 0]@
         Exp_{QU_1,$k}[η, Δ_{1→1,2}[QU[y_1]] /. QU → Times],
        ReplacePart[2 → 0]@
         Exp_{QU_3,$k}[α, Δ_{3→3,4}[QU[a_3]] /. QU → Times],
        ReplacePart[1 → 0]@
         Exp_{QU_5,$k}[ξ, Δ_{5→5,6}[QU[x_5]] /. QU → Times]
       ] /. {η → η_1, α → α_1, ξ → ξ_1}];
   𝔼[τ_1 (t_1 + t_2) + α_1 (a_1 + a_2), OE⟦2⟧, OE⟦3⟧]];
tΔ_{i_→j_,k_} :=
  Δ[$U, $k] /. {(v : τ | η | α | ξ)_1 → v_i,
    (v : t | T | y | a | x)_1 → v_j, (v : t | T | y | a | x)_2 → v_k};
```

```
tΔ_{1→1,2}
```

$$\mathbb{E}\Big[(a_1+a_2)\,\alpha_1 + (t_1+t_2)\,\tau_1,\ y_1\,\eta_1 + T_1\,y_2\,\eta_1 + x_1\,\xi_1 + x_2\,\xi_1,$$
$$1 + \frac{1}{2}\,\big(-2\hbar\,a_1\,T_1\,y_2\,\eta_1 + \gamma\,\hbar\,T_1\,y_1\,y_2\,\eta_1^2 - 2\hbar\,a_1\,x_2\,\xi_1 + \gamma\,\hbar\,x_1\,x_2\,\xi_1^2\big)\,\epsilon + O[\epsilon]^2\Big]$$

The Faddeev-Quesne formula:

```
e_{q_,k_}[x_] := e^(∑_{j=1}^{k+1} (1 - q)^j x^j / (j (1 - q^j))); e_{q_}[x_] := e_{q,$k}[x]
```

```
R[QU, kk_] :=
  R[QU, kk] = 𝔼[-ℏ a_2 t_1/γ, ℏ x_2 y_1,
    Series[e^{ℏ γ^-1 t_1 a_2 - ℏ y_1 x_2}
      (e^{ℏ b_1 a_2} e_{q_ℏ,kk}[ℏ y_1 x_2] /. b_1 → γ^-1 (ϵ a_1 - t_1)),
     {ϵ, 0, kk}]];
tR_{i_,j_} :=
  R[$U, $k] /. {(v : t | T | y | a | x)_1 → v_i,
    (v : t | T | y | a | x)_2 → v_j};
t̄R_{i_,j_} := t̄R_{i,j} = tR_{i,j} ~B_j~ tS_j;
```

```
{tR_{1,2}, t̄R_{1,2}}
```

$$\Big\{\mathbb{E}\Big[-\frac{\hbar\,a_2\,t_1}{\gamma},\ \hbar\,x_2\,y_1,\ 1 + \Big(\frac{\hbar\,a_1\,a_2}{\gamma} - \frac{1}{4}\,\gamma\,\hbar^3\,x_2^2\,y_1^2\Big)\,\epsilon + O[\epsilon]^2\Big],$$
$$\mathbb{E}\Big[\frac{\hbar\,a_2\,t_1}{\gamma},\ -\frac{\hbar\,x_2\,y_1}{T_1},\ 1 + \frac{1}{4\,\gamma\,T_1^2}\,\big(-4\hbar\,a_1\,a_2\,T_1^2 - 4\,\gamma\,\hbar^2\,a_1\,T_1\,x_2\,y_1 - 4\,\gamma\,\hbar^2\,a_2\,T_1\,x_2\,y_1 - 3\,\gamma^2\,\hbar^3\,x_2^2\,y_1^2\big)\,\epsilon + O[\epsilon]^2\Big]\Big\}$$

tC is the counterclockwise spinner; t̄C is its inverse.

```
tC_i_ := 𝔼[0, 0, T_i^{1/2} e^{-ϵ a_i ℏ} + 0_{$k}];
t̄C_i_ := 𝔼[0, 0, T_i^{-1/2} e^{ϵ a_i ℏ} + 0_{$k}];
```

```
Block[{$k = 3}, {tC_1, t̄C_2}]
```

$$\Big\{\mathbb{E}\Big[0,\ 0,\ \sqrt{T_1} - \hbar\,a_1\,\sqrt{T_1}\,\epsilon + \frac{1}{2}\,\hbar^2\,a_1^2\,\sqrt{T_1}\,\epsilon^2 - \frac{1}{6}\,\big(\hbar^3\,a_1^3\,\sqrt{T_1}\big)\,\epsilon^3 + O[\epsilon]^4\Big],$$
$$\mathbb{E}\Big[0,\ 0,\ \frac{1}{\sqrt{T_2}} + \frac{\hbar\,a_2\,\epsilon}{\sqrt{T_2}} + \frac{\hbar^2\,a_2^2\,\epsilon^2}{2\sqrt{T_2}} + \frac{\hbar^3\,a_2^3\,\epsilon^3}{6\sqrt{T_2}} + O[\epsilon]^4\Big]\Big\}$$

```
Kink[QU, kk_] :=
  Kink[QU, kk] =
   Block[{$k = kk}, (tR_{1,3} t̄C_2) ~B_{1,2}~ tm_{1,2→1} ~B_{1,3}~ tm_{1,3→1}];
tKink_i_ := Kink[$U, $k] /. {(v : t | T | y | a | x)_1 → v_i};
K̄ink[QU, kk_] :=
  K̄ink[QU, kk] =
   Block[{$k = kk}, (t̄R_{1,3} tC_2) ~B_{1,2}~ tm_{1,2→1} ~B_{1,3}~ tm_{1,3→1}];
t̄Kink_i_ := K̄ink[$U, $k] /. {(v : t | T | y | a | x)_1 → v_i}
```

Alternative Algorithms

```
λ_{alt,k_}[CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
   eq = ρ@e^{ξ xCU}.ρ@e^{η yCU} == ρ@e^{d yCU}.ρ@e^{c (t 1CU - 2 ϵ aCU)}.ρ@e^{b xCU};
   {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /.
     C@1 → 0;
   Series[e^{-η y - ξ x + η ξ t + c t + d y - 2 ϵ c a + b x} /. so, {ϵ, 0, k}]]];
```

The Trefoil

```
Block[{$k = 1},
 Z = tR_{1,5} tR_{6,2} tR_{3,7} t̄C_4 t̄Kink_8 t̄Kink_9 t̄Kink_10;
 Do[Z = Z ~B_{1,k}~ tm_{1,k→1}, {k, 2, 10}]; Z]
```

$$\mathbb{E}\Big[0,\ 0,\ \frac{T_1}{1 - T_1 + T_1^2} + \Big(\big(-2\hbar\,a_1\,T_1 - \gamma\,\hbar\,T_1^2 + 2\hbar\,a_1\,T_1^2 + 2\,\gamma\,\hbar\,T_1^3 - 3\,\gamma\,\hbar\,T_1^4 - 2\hbar\,a_1\,T_1^4 + 2\,\gamma\,\hbar\,T_1^5 + 2\hbar\,a_1\,T_1^5 - 2\,\gamma\,\hbar^2\,T_1\,x_1\,y_1 - 2\,\gamma\,\hbar^2\,T_1^4\,x_1\,y_1\big)\,\epsilon\Big) \Big/ \big(1 - 3\,T_1 + 6\,T_1^2 - 7\,T_1^3 + 6\,T_1^4 - 3\,T_1^5 + T_1^6\big) + O[\epsilon]^2\Big]$$

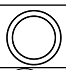
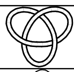
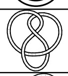
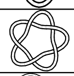
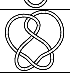
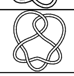
diagram	n_k^t Alexander's ω^+ / Today's / Rozansky's ρ_1^+	genus / ribbon / unknotting number / amphicheiral	diagram	n_k^t Alexander's ω^+ / Today's / Rozansky's ρ_1^+	genus / ribbon / unknotting number / amphicheiral
	0_1^a 1 0	0 / ✔ 0 / ✔		3_1^a $t-1$ t	1 / ✘ 1 / ✘
	4_1^a $3-t$ 0	1 / ✘ 1 / ✔		5_1^a t^2-t+1 $2t^3+3t$	2 / ✘ 2 / ✘
	5_2^a $2t-3$ $5t-4$	1 / ✘ 1 / ✘		6_1^a $5-2t$ $t-4$	1 / ✔ 1 / ✘

diagram	n_k^t	Alexander's ω^+	Today's / Rozansky's ρ_1^+	genus / ribbon	unknotting number / amphicheiral
	6_2^a	$-t^2+3t-3$	t^3-4t^2+4t-4	2 / ✘	1 / ✘
	6_3^a	t^2-3t+5	0	2 / ✘	1 / ✔
	7_1^a	t^3-t^2+t-1	$3t^5+5t^3+6t$	3 / ✘	3 / ✘
	7_2^a	$3t-5$	$14t-16$	1 / ✘	1 / ✘
	7_3^a	$2t^2-3t+3$	$-9t^3+8t^2-16t+12$	2 / ✘	2 / ✘
	7_4^a	$4t-7$	$32-24t$	1 / ✘	2 / ✘
	7_5^a	$2t^2-4t+5$	$9t^3-16t^2+29t-28$	2 / ✘	2 / ✘
	7_6^a	$-t^2+5t-7$	$t^3-8t^2+19t-20$	2 / ✘	1 / ✘
	7_7^a	t^2-5t+9	$8-3t$	2 / ✘	1 / ✘
	8_1^a	$7-3t$	$5t-16$	1 / ✘	1 / ✘
	8_2^a	$-t^3+3t^2-3t+3$	$2t^5-8t^4+10t^3-12t^2+13t-12$	3 / ✘	2 / ✘
	8_3^a	$9-4t$	0	1 / ✘	2 / ✔
	8_4^a	$-2t^2+5t-5$	$3t^3-8t^2+6t-4$	2 / ✘	2 / ✘
	8_5^a	$-t^3+3t^2-4t+5$	$-2t^5+8t^4-13t^3+20t^2-22t+24$	3 / ✘	2 / ✘
	8_6^a	$-2t^2+6t-7$	$5t^3-20t^2+28t-32$	2 / ✘	2 / ✘
	8_7^a	t^3-3t^2+5t-5	$-t^5+4t^4-10t^3+12t^2-13t+12$	3 / ✘	1 / ✘
	8_8^a	$2t^2-6t+9$	$-t^3+4t^2-12t+16$	2 / ✔	2 / ✘
	8_9^a	$-t^3+3t^2-5t+7$	0	3 / ✔	1 / ✔
	8_{10}^a	t^3-3t^2+6t-7	$-t^5+4t^4-11t^3+16t^2-21t+20$	3 / ✘	2 / ✘
	8_{11}^a	$-2t^2+7t-9$	$5t^3-24t^2+39t-44$	2 / ✘	1 / ✘
	8_{12}^a	$t^2-7t+13$	0	2 / ✘	2 / ✔
	8_{13}^a	$2t^2-7t+11$	$-t^3+4t^2-14t+20$	2 / ✘	1 / ✘
	8_{14}^a	$-2t^2+8t-11$	$5t^3-28t^2+57t-68$	2 / ✘	1 / ✘
	8_{15}^a	$3t^2-8t+11$	$21t^3-64t^2+120t-140$	2 / ✘	2 / ✘
	8_{16}^a	t^3-4t^2+8t-9	$t^5-6t^4+17t^3-28t^2+35t-36$	3 / ✘	2 / ✘
	8_{17}^a	$-t^3+4t^2-8t+11$	0	3 / ✘	1 / ✔
	8_{18}^a	$-t^3+5t^2-10t+13$	0	3 / ✘	2 / ✔
	8_{19}^n	t^3-t^2+1	$-3t^5-4t^2-3t$	3 / ✘	3 / ✘
	8_{20}^n	t^2-2t+3	$4t-4$	2 / ✔	1 / ✘
	8_{21}^n	$-t^2+4t-5$	$t^3-8t^2+16t-20$	2 / ✘	1 / ✘
	9_1^a	$t^4-t^3+t^2-t+1$	$4t^7+7t^5+9t^3+10t$	4 / ✘	4 / ✘
	9_2^a	$4t-7$	$30t-40$	1 / ✘	1 / ✘
	9_3^a	$2t^3-3t^2+3t-3$	$-13t^5+12t^4-25t^3+20t^2-32t+24$	3 / ✘	3 / ✘
	9_4^a	$3t^2-5t+5$	$23t^3-28t^2+46t-44$	2 / ✘	2 / ✘
	9_5^a	$6t-11$	$100-65t$	1 / ✘	2 / ✘
	9_6^a	$2t^3-4t^2+5t-5$	$13t^5-24t^4+45t^3-52t^2+68t-64$	3 / ✘	3 / ✘
	9_7^a	$3t^2-7t+9$	$23t^3-56t^2+99t-108$	2 / ✘	2 / ✘
	9_8^a	$-2t^2+8t-11$	$3t^3-16t^2+29t-28$	2 / ✘	2 / ✘
	9_9^a	$2t^3-4t^2+6t-7$	$13t^5-24t^4+55t^3-72t^2+98t-96$	3 / ✘	3 / ✘
	9_{10}^a	$4t^2-8t+9$	$-40t^3+72t^2-114t+120$	2 / ✘	2, 3 / ✘
	9_{11}^a	$-t^3+5t^2-7t+7$	$-2t^5+16t^4-41t^3+52t^2-66t+64$	3 / ✘	2 / ✘
	9_{12}^a	$-2t^2+9t-13$	$5t^3-36t^2+84t-100$	2 / ✘	1 / ✘
	9_{13}^a	$4t^2-9t+11$	$-40t^3+92t^2-154t+168$	2 / ✘	2, 3 / ✘
	9_{14}^a	$2t^2-9t+15$	$-t^3+8t^2-35t+60$	2 / ✘	1 / ✘
	9_{15}^a	$-2t^2+10t-15$	$-5t^3+40t^2-108t+136$	2 / ✘	2 / ✘
	9_{16}^a	$2t^3-5t^2+8t-9$	$-13t^5+36t^4-80t^3+120t^2-161t+168$	3 / ✘	3 / ✘
	9_{17}^a	t^3-5t^2+9t-9	$t^5-8t^4+23t^3-32t^2+28t-24$	3 / ✘	2 / ✘
	9_{18}^a	$4t^2-10t+13$	$40t^3-108t^2+193t-220$	2 / ✘	2 / ✘
	9_{19}^a	$2t^2-10t+17$	$t^3-8t^2+20t-24$	2 / ✘	1 / ✘
	9_{20}^a	$-t^3+5t^2-9t+11$	$2t^5-16t^4+47t^3-84t^2+117t-124$	3 / ✘	2 / ✘
	9_{21}^a	$-2t^2+11t-17$	$-5t^3+44t^2-127t+164$	2 / ✘	1 / ✘
	9_{22}^a	$t^3-5t^2+10t-11$	$-t^5+8t^4-24t^3+38t^2-40t+36$	3 / ✘	1 / ✘
	9_{23}^a	$4t^2-11t+15$	$40t^3-128t^2+243t-288$	2 / ✘	2 / ✘
	9_{24}^a	$-t^3+5t^2-10t+13$	$-4t^2+16t-20$	3 / ✘	1 / ✘
	9_{25}^a	$-3t^2+12t-17$	$12t^3-70t^2+153t-188$	2 / ✘	2 / ✘
	9_{26}^a	$t^3-5t^2+11t-13$	$-t^5+8t^4-31t^3+64t^2-85t+92$	3 / ✘	1 / ✘
	9_{27}^a	$-t^3+5t^2-11t+15$	$t^3-8t^2+24t-32$	3 / ✔	1 / ✘
	9_{28}^a	$t^3-5t^2+12t-15$	$t^5-8t^4+30t^3-68t^2+105t-120$	3 / ✘	1 / ✘
	9_{29}^a	$t^3-5t^2+12t-15$	$t^5-8t^4+26t^3-48t^2+59t-56$	3 / ✘	2 / ✘
	9_{30}^a	$-t^3+5t^2-12t+17$	$2t^3-10t^2+25t-32$	3 / ✘	1 / ✘
	9_{31}^a	$t^3-5t^2+13t-17$	$t^5-8t^4+33t^3-80t^2+132t-152$	3 / ✘	2 / ✘
	9_{32}^a	$t^3-6t^2+14t-17$	$-t^5+10t^4-42t^3+94t^2-133t+148$	3 / ✘	2 / ✘
	9_{33}^a	$-t^3+6t^2-14t+19$	$t^3-10t^2+30t-40$	3 / ✘	1 / ✘
	9_{34}^a	$-t^3+6t^2-16t+23$	$3t^3-18t^2+43t-56$	3 / ✘	1 / ✘

n^t_k	Alexander's ω^+	Today's / Rozansky's ρ_1^+	genus / ribbon	unknotting number / amphicheiral	n^t_k	Alexander's ω^+	Today's / Rozansky's ρ_1^+	genus / ribbon	unknotting number / amphicheiral
9^a_{35}	$7t-13$	$90t-144$	1 / ✘	2, 3 / ✘	9^a_{36}	$-t^3+5t^2-8t+9$	$-2t^5+16t^4-44t^3+66t^2-87t+88$	3 / ✘	2 / ✘
9^a_{37}	$2t^2-11t+19$	$t^3-8t^2+22t-28$	2 / ✘	2 / ✘	9^a_{38}	$5t^2-14t+19$	$62t^3-204t^2+382t-452$	2 / ✘	2, 3 / ✘
9^a_{39}	$-3t^2+14t-21$	$-12t^3+84t^2-210t+268$	2 / ✘	1 / ✘	9^a_{40}	$t^3-7t^2+18t-23$	$t^5-12t^4+57t^3-144t^2+229t-264$	3 / ✘	2 / ✘
9^a_{41}	$3t^2-12t+19$	$3t^3-20t^2+70t-108$	2 / ✔	2 / ✘	9^n_{42}	$-t^2+2t-1$	$-t^3+2t^2+t-4$	2 / ✘	1 / ✘
9^n_{43}	$-t^3+3t^2-2t+1$	$-2t^5+8t^4-7t^3+2t^2-5t+4$	3 / ✘	2 / ✘	9^n_{44}	t^2-4t+7	$-2t^2+9t-12$	2 / ✘	1 / ✘
9^n_{45}	$-t^2+6t-9$	$t^3-14t^2+47t-60$	2 / ✘	1 / ✘	9^n_{46}	$5-2t$	$3t-12$	1 / ✔	2 / ✘
9^n_{47}	t^3-4t^2+6t-5	$-t^5+6t^4-15t^3+16t^2-10t+12$	3 / ✘	2 / ✘	9^n_{48}	$-t^2+7t-11$	$-t^3+12t^2-42t+52$	2 / ✘	2 / ✘
9^n_{49}	$3t^2-6t+7$	$-21t^3+38t^2-61t+60$	2 / ✘	3 / ✘	10^a_{1}	$9-4t$	$14t-40$	1 / ✘	1 / ✘
10^a_{2}	$-t^4+3t^3-3t^2+3t-3$	$3t^7-12t^6+16t^5-20t^4+24t^3-24t^2+27t-24$	4 / ✘	3 / ✘	10^a_{3}	$13-6t$	$11t-28$	1 / ✔	2 / ✘
10^a_{4}	$-3t^2+7t-7$	$4t^3-8t^2+t+8$	2 / ✘	2 / ✘	10^a_{5}	$t^4-3t^3+5t^2-5t+5$	$-2t^7+8t^6-20t^5+28t^4-36t^3+36t^2-39t+36$	4 / ✘	2 / ✘
10^a_{6}	$-2t^3+6t^2-7t+7$	$9t^5-36t^4+56t^3-72t^2+81t-84$	3 / ✘	3 / ✘	10^a_{7}	$-3t^2+11t-15$	$14t^3-72t^2+135t-160$	2 / ✘	1 / ✘
10^a_{8}	$-2t^3+5t^2-5t+5$	$7t^5-20t^4+23t^3-28t^2+26t-24$	3 / ✘	2 / ✘	10^a_{9}	$-t^4+3t^3-5t^2+7t-7$	$-t^7+4t^6-10t^5+20t^4-25t^3+28t^2-28t+28$	4 / ✘	1 / ✘
10^a_{10}	$3t^2-11t+17$	$-5t^3+24t^2-71t+100$	2 / ✘	1 / ✘	10^a_{11}	$-4t^2+11t-13$	$16t^3-52t^2+68t-72$	2 / ✘	2, 3 / ✘
10^a_{12}	$2t^3-6t^2+10t-11$	$-5t^5+20t^4-50t^3+72t^2-89t+92$	3 / ✘	2 / ✘	10^a_{13}	$2t^2-13t+23$	$t^3-12t^2+51t-84$	2 / ✘	2 / ✘
10^a_{14}	$-2t^3+8t^2-12t+13$	$9t^5-52t^4+119t^3-180t^2+225t-236$	3 / ✘	2 / ✘	10^a_{15}	$2t^3-6t^2+9t-9$	$-3t^5+12t^4-24t^3+24t^2-17t+12$	3 / ✘	2 / ✘
10^a_{16}	$-4t^2+12t-15$	$-16t^3+56t^2-76t+80$	2 / ✘	2 / ✘	10^a_{17}	$t^4-3t^3+5t^2-7t+9$	0	4 / ✘	1 / ✔
10^a_{18}	$-4t^2+14t-19$	$16t^3-68t^2+121t-140$	2 / ✘	1 / ✘	10^a_{19}	$2t^3-7t^2+11t-11$	$3t^5-16t^4+35t^3-40t^2+30t-24$	3 / ✘	2 / ✘
10^a_{20}	$-3t^2+9t-11$	$14t^3-56t^2+88t-104$	2 / ✘	2 / ✘	10^a_{21}	$-2t^3+7t^2-9t+9$	$9t^5-44t^4+80t^3-104t^2+121t-124$	3 / ✘	2 / ✘
10^a_{22}	$-2t^3+6t^2-10t+13$	$-t^5+4t^4-10t^3+24t^2-37t+44$	3 / ✔	2 / ✘	10^a_{23}	$2t^3-7t^2+13t-15$	$-5t^5+24t^4-67t^3+108t^2-137t+144$	3 / ✘	1 / ✘
10^a_{24}	$-4t^2+14t-19$	$24t^3-116t^2+221t-268$	2 / ✘	2 / ✘	10^a_{25}	$-2t^3+8t^2-14t+17$	$9t^5-52t^4+131t^3-232t^2+314t-344$	3 / ✘	2 / ✘
10^a_{26}	$-2t^3+7t^2-13t+17$	$-t^5+4t^4-10t^3+28t^2-49t+60$	3 / ✘	1 / ✘	10^a_{27}	$2t^3-8t^2+16t-19$	$5t^5-28t^4+87t^3-164t^2+229t-252$	3 / ✘	1 / ✘
10^a_{28}	$4t^2-13t+19$	$-8t^3+36t^2-100t+136$	2 / ✘	2 / ✘	10^a_{29}	$t^3-7t^2+15t-17$	$t^5-12t^4+52t^3-104t^2+124t-128$	3 / ✘	2 / ✘
10^a_{30}	$-4t^2+17t-25$	$24t^3-148t^2+345t-440$	2 / ✘	1 / ✘	10^a_{31}	$4t^2-14t+21$	$-4t^2+9t-12$	2 / ✘	1 / ✘
10^a_{32}	$-2t^3+8t^2-15t+19$	$t^5-4t^4+13t^3-40t^2+78t-96$	3 / ✘	1 / ✘	10^a_{33}	$4t^2-16t+25$	0	2 / ✘	1 / ✔
10^a_{34}	$3t^2-9t+13$	$-5t^3+20t^2-52t+68$	2 / ✘	2 / ✘	10^a_{35}	$2t^2-12t+21$	$-t^3+12t^2-47t+76$	2 / ✔	2 / ✘
10^a_{36}	$-3t^2+13t-19$	$14t^3-88t^2+208t-264$	2 / ✘	2 / ✘	10^a_{37}	$4t^2-13t+19$	0	2 / ✘	2 / ✔
10^a_{38}	$-4t^2+15t-21$	$24t^3-128t^2+270t-336$	2 / ✘	2 / ✘	10^a_{39}	$-2t^3+8t^2-13t+15$	$9t^5-52t^4+125t^3-204t^2+263t-280$	3 / ✘	2 / ✘
10^a_{40}	$2t^3-8t^2+17t-21$	$-5t^5+28t^4-89t^3+176t^2-258t+288$	3 / ✘	2 / ✘	10^a_{41}	$t^3-7t^2+17t-21$	$t^5-12t^4+54t^3-120t^2+157t-164$	3 / ✘	2 / ✘
10^a_{42}	$-t^3+7t^2-19t+27$	$2t^3-8t^2+11t-12$	3 / ✔	1 / ✘	10^a_{43}	$-t^3+7t^2-17t+23$	0	3 / ✘	2 / ✔
10^a_{44}	$t^3-7t^2+19t-25$	$t^5-12t^4+56t^3-140t^2+220t-248$	3 / ✘	1 / ✘	10^a_{45}	$-t^3+7t^2-21t+31$	0	3 / ✘	2 / ✔
10^a_{46}	$-t^4+3t^3-4t^2+5t-5$	$-3t^7+12t^6-21t^5+34t^4-43t^3+52t^2-55t+56$	4 / ✘	3 / ✘	10^a_{47}	$t^4-3t^3+6t^2-7t+7$	$-2t^7+8t^6-23t^5+38t^4-56t^3+60t^2-68t+64$	4 / ✘	2, 3 / ✘
10^a_{48}	$t^4-3t^3+6t^2-9t+11$	$t^5-2t^4+2t^3-3t+4$	4 / ✔	2 / ✘	10^a_{49}	$3t^3-8t^2+12t-13$	$30t^5-94t^4+196t^3-292t^2+372t-392$	3 / ✘	3 / ✘

n^t_k	Alexander's ω^+	Today's / Rozansky's ρ_1^+	genus / ribbon	unknotting number / amphicheiral
10^a_{50}	$-2t^3+7t^2-11t+13$	$-9t^5+44t^4-94t^3+150t^2-186t+200$	3 / ✘	2 / ✘
10^a_{51}	$2t^3-7t^2+15t-19$	$-5t^5+24t^4-73t^3+134t^2-194t+212$	3 / ✘	2, 3 / ✘
10^a_{52}	$2t^3-7t^2+13t-15$	$-3t^5+16t^4-37t^3+50t^2-49t+44$	3 / ✘	2 / ✘
10^a_{53}	$6t^2-18t+25$	$93t^3-346t^2+680t-828$	2 / ✘	2, 3 / ✘
10^a_{54}	$2t^3-6t^2+10t-11$	$-3t^5+12t^4-24t^3+26t^2-21t+16$	3 / ✘	2, 3 / ✘
10^a_{55}	$5t^2-15t+21$	$66t^3-246t^2+488t-596$	2 / ✘	2 / ✘
10^a_{56}	$-2t^3+8t^2-14t+17$	$-9t^5+52t^4-133t^3+234t^2-312t+340$	3 / ✘	2 / ✘
10^a_{57}	$2t^3-8t^2+18t-23$	$-5t^5+28t^4-93t^3+194t^2-300t+340$	3 / ✘	2 / ✘
10^a_{58}	$3t^2-16t+27$	$3t^3-28t^2+94t-140$	2 / ✘	2 / ✘
10^a_{59}	$t^3-7t^2+18t-23$	$-t^5+12t^4-55t^3+128t^2-181t+196$	3 / ✘	1 / ✘
10^a_{60}	$-t^3+7t^2-20t+29$	$5t^3-40t^2+122t-176$	3 / ✘	1 / ✘
10^a_{61}	$-2t^3+5t^2-6t+7$	$-7t^5+20t^4-27t^3+36t^2-35t+36$	3 / ✘	2, 3 / ✘
10^a_{62}	$t^4-3t^3+6t^2-8t+9$	$-2t^7+8t^6-23t^5+40t^4-63t^3+76t^2-89t+88$	4 / ✘	2 / ✘
10^a_{63}	$5t^2-14t+19$	$66t^3-220t^2+416t-496$	2 / ✘	2 / ✘
10^a_{64}	$-t^4+3t^3-6t^2+10t-11$	$-t^7+4t^6-11t^5+24t^4-37t^3+52t^2-60t+64$	4 / ✘	2 / ✘
10^a_{65}	$2t^3-7t^2+14t-17$	$-5t^5+24t^4-71t^3+124t^2-169t+180$	3 / ✘	2 / ✘
10^a_{66}	$3t^3-9t^2+16t-19$	$30t^5-112t^4+279t^3-480t^2+662t-724$	3 / ✘	3 / ✘
10^a_{67}	$-4t^2+16t-23$	$24t^3-140t^2+312t-392$	2 / ✘	2 / ✘
10^a_{68}	$4t^2-14t+21$	$8t^3-40t^2+117t-164$	2 / ✘	2 / ✘
10^a_{69}	$t^3-7t^2+21t-29$	$-t^5+12t^4-68t^3+212t^2-397t+476$	3 / ✘	2 / ✘
10^a_{70}	$t^3-7t^2+16t-19$	$-t^5+12t^4-53t^3+114t^2-146t+152$	3 / ✘	2 / ✘
10^a_{71}	$-t^3+7t^2-18t+25$	t^3-2t^2-t+4	3 / ✘	1 / ✘
10^a_{72}	$-2t^3+9t^2-16t+19$	$-9t^5+60t^4-167t^3+298t^2-410t+448$	3 / ✘	2 / ✘
10^a_{73}	$t^3-7t^2+20t-27$	$t^5-12t^4+65t^3-194t^2+350t-416$	3 / ✘	1 / ✘
10^a_{74}	$-4t^2+16t-23$	$24t^3-136t^2+290t-360$	2 / ✘	2 / ✘
10^a_{75}	$-t^3+7t^2-19t+27$	$-4t^3+36t^2-117t+172$	3 / ✔	2 / ✘
10^a_{76}	$-2t^3+7t^2-12t+15$	$-9t^5+44t^4-104t^3+184t^2-245t+272$	3 / ✘	2, 3 / ✘
10^a_{77}	$2t^3-7t^2+14t-17$	$-5t^5+24t^4-71t^3+132t^2-189t+208$	3 / ✘	2, 3 / ✘
10^a_{78}	$-t^3+7t^2-16t+21$	$2t^5-24t^4+105t^3-244t^2+390t-448$	3 / ✘	2 / ✘
10^a_{79}	$t^4-3t^3+7t^2-12t+15$	0	4 / ✘	2, 3 / ✔
10^a_{80}	$3t^3-9t^2+15t-17$	$30t^5-112t^4+260t^3-426t^2+568t-616$	3 / ✘	3 / ✘
10^a_{81}	$-t^3+8t^2-20t+27$	0	3 / ✘	2 / ✔
10^a_{82}	$-t^4+4t^3-8t^2+12t-13$	$t^7-6t^6+19t^5-42t^4+64t^3-78t^2+84t-84$	4 / ✘	1 / ✘
10^a_{83}	$2t^3-9t^2+19t-23$	$-5t^5+34t^4-110t^3+214t^2-301t+332$	3 / ✘	2 / ✘
10^a_{84}	$2t^3-9t^2+20t-25$	$-5t^5+34t^4-116t^3+246t^2-373t+424$	3 / ✘	1 / ✘
10^a_{85}	$t^4-4t^3+8t^2-10t+11$	$2t^7-12t^6+36t^5-68t^4+101t^3-124t^2+138t-140$	4 / ✘	2 / ✘
10^a_{86}	$-2t^3+9t^2-19t+25$	$-t^5+6t^4-21t^3+58t^2-105t+128$	3 / ✘	2 / ✘
10^a_{87}	$-2t^3+9t^2-18t+23$	$-t^5+6t^4-23t^3+66t^2-125t+152$	3 / ✔	2 / ✘
10^a_{88}	$-t^3+8t^2-24t+35$	0	3 / ✘	1 / ✔
10^a_{89}	$t^3-8t^2+24t-33$	$t^5-14t^4+83t^3-264t^2+495t-596$	3 / ✘	2 / ✘
10^a_{90}	$-2t^3+8t^2-17t+23$	$-t^5+6t^4-21t^3+54t^2-93t+112$	3 / ✘	2 / ✘
10^a_{91}	$t^4-4t^3+9t^2-14t+17$	$t^5-2t^4+2t^3-3t+4$	4 / ✘	1 / ✘
10^a_{92}	$-2t^3+10t^2-20t+25$	$-9t^5+68t^4-216t^3+428t^2-622t+696$	3 / ✘	2 / ✘
10^a_{93}	$2t^3-8t^2+15t-17$	$3t^5-18t^4+43t^3-58t^2+55t-48$	3 / ✘	2 / ✘
10^a_{94}	$-t^4+4t^3-9t^2+14t-15$	$-t^7+6t^6-20t^5+46t^4-76t^3+102t^2-115t+120$	4 / ✘	2 / ✘
10^a_{95}	$2t^3-9t^2+21t-27$	$-5t^5+32t^4-114t^3+248t^2-384t+436$	3 / ✘	1 / ✘
10^a_{96}	$-t^3+7t^2-22t+33$	$-7t^3+50t^2-147t+212$	3 / ✘	2 / ✘
10^a_{97}	$-5t^2+22t-33$	$-37t^3+242t^2-603t+788$	2 / ✘	2 / ✘
10^a_{98}	$-2t^3+9t^2-18t+23$	$9t^5-60t^4+177t^3-348t^2+501t-564$	3 / ✘	2 / ✘
10^a_{99}	$t^4-4t^3+10t^2-16t+19$	0	4 / ✔	2 / ✔
10^a_{100}	$t^4-4t^3+9t^2-12t+13$	$2t^7-12t^6+39t^5-80t^4+128t^3-164t^2+192t-196$	4 / ✘	2, 3 / ✘
10^a_{101}	$7t^2-21t+29$	$-129t^3+480t^2-942t+1148$	2 / ✘	2, 3 / ✘
10^a_{102}	$-2t^3+8t^2-16t+21$	$-t^5+6t^4-19t^3+50t^2-89t+108$	3 / ✘	1 / ✘
10^a_{103}	$2t^3-8t^2+17t-21$	$5t^5-30t^4+93t^3-178t^2+254t-280$	3 / ✘	3 / ✘
10^a_{104}	$t^4-4t^3+9t^2-15t+19$	$t^5-2t^4+2t^3-3t+4$	4 / ✘	1 / ✘
10^a_{105}	$t^3-8t^2+22t-29$	$-t^5+14t^4-71t^3+184t^2-292t+332$	3 / ✘	2 / ✘
10^a_{106}	$-t^4+4t^3-9t^2+15t-17$	$-t^7+6t^6-20t^5+48t^4-82t^3+114t^2-134t+140$	4 / ✘	2 / ✘
10^a_{107}	$-t^3+8t^2-22t+31$	$2t^3-8t^2+13t-16$	3 / ✘	1 / ✘
10^a_{108}	$2t^3-8t^2+14t-15$	$-3t^5+18t^4-41t^3+50t^2-40t+32$	3 / ✘	2 / ✘
10^a_{109}	$t^4-4t^3+10t^2-17t+21$	0	4 / ✘	2 / ✔
10^a_{110}	$t^3-8t^2+20t-25$	$t^5-14t^4+69t^3-160t^2+219t-236$	3 / ✘	2 / ✘
10^a_{111}	$-2t^3+9t^2-17t+21$	$-9t^5+60t^4-171t^3+316t^2-436t+480$	3 / ✘	2 / ✘
10^a_{112}	$-t^4+5t^3-11t^2+17t-19$	$t^7-8t^6+29t^5-68t^4+115t^3-152t^2+175t-180$	4 / ✘	2 / ✘
10^a_{113}	$2t^3-11t^2+26t-33$	$-5t^5+42t^4-167t^3+394t^2-623t+720$	3 / ✘	1 / ✘

diagram	n_k^t	Alexander's ω^+	genus / ribbon	Today's / Rozansky's ρ_1^+	unknotting number / amphicheiral
	10^a_{114}	$-2t^3+10t^2-21t+27$	3 / ✘	$t^5-8t^4+30t^3-78t^2+140t-168$	1 / ✘
	10^a_{115}	$-t^3+9t^2-26t+37$	3 / ✘	0	2 / ✔
	10^a_{116}	$-t^4+5t^3-12t^2+19t-21$	4 / ✘	$t^7-8t^6+30t^5-74t^4+132t^3-184t^2+217t-228$	2 / ✘
	10^a_{117}	$2t^3-10t^2+24t-31$	3 / ✘	$-5t^5+38t^4-144t^3+330t^2-522t+600$	2 / ✘
	10^a_{118}	$t^4-5t^3+12t^2-19t+23$	4 / ✘	0	1 / ✔
	10^a_{119}	$-2t^3+10t^2-23t+31$	3 / ✘	$-t^5+6t^4-26t^3+86t^2-175t+220$	1 / ✘
	10^a_{120}	$8t^2-26t+37$	2 / ✘	$166t^3-692t^2+1433t-1788$	2, 3 / ✘
	10^a_{121}	$2t^3-11t^2+27t-35$	3 / ✘	$5t^5-42t^4+167t^3-396t^2+634t-732$	2 / ✘
	10^a_{122}	$-2t^3+11t^2-24t+31$	3 / ✘	$-t^5+8t^4-34t^3+104t^2-211t+264$	2 / ✘
	10^a_{123}	$t^4-6t^3+15t^2-24t+29$	4 / ✔	0	2 / ✔
	10^n_{124}	t^4-t^3+t-1	4 / ✘	$-4t^7-6t^4-4t^2-6t$	4 / ✘
	10^n_{125}	t^3-2t^2+2t-1	3 / ✘	$-t^5+2t^4-2t^3+3t-4$	2 / ✘
	10^n_{126}	t^3-2t^2+4t-5	3 / ✘	$t^5-2t^4+10t^3-12t^2+22t-20$	2 / ✘
	10^n_{127}	$-t^3+4t^2-6t+7$	3 / ✘	$2t^5-14t^4+32t^3-52t^2+67t-72$	2 / ✘
	10^n_{128}	$2t^3-3t^2+t+1$	3 / ✘	$-13t^5+12t^4-3t^3-10t^2-9t+12$	3 / ✘
	10^n_{129}	$2t^2-6t+9$	2 / ✔	$-t^3-2t^2+14t-20$	1 / ✘
	10^n_{130}	$2t^2-4t+5$	2 / ✘	$t^3-2t^2+19t-24$	2 / ✘
	10^n_{131}	$-2t^2+8t-11$	2 / ✘	$5t^3-38t^2+87t-112$	1 / ✘
	10^n_{132}	t^2-t+1	2 / ✘	$2t^2+5t-4$	1 / ✘
	10^n_{133}	$-t^2+5t-7$	2 / ✘	$t^3-14t^2+37t-48$	1 / ✘
	10^n_{134}	$2t^3-4t^2+4t-3$	3 / ✘	$-13t^5+24t^4-33t^3+30t^2-41t+40$	3 / ✘
	10^n_{135}	$3t^2-9t+13$	2 / ✘	$t^3-6t^2+18t-24$	2 / ✘
	10^n_{136}	$-t^2+4t-5$	2 / ✘	$-t^3+4t^2-2t-4$	1 / ✘
	10^n_{137}	$t^2-6t+11$	2 / ✔	$-4t^2+24t-44$	1 / ✘
	10^n_{138}	t^3-5t^2+8t-7	3 / ✘	$-t^5+8t^4-22t^3+24t^2-11t+8$	2 / ✘
	10^n_{139}	t^4-t^3+2t-3	4 / ✘	$-4t^7-12t^4+5t^3-4t^2-16t+12$	4 / ✘
	10^n_{140}	t^2-2t+3	2 / ✔	$8t-8$	2 / ✘
	10^n_{141}	$-t^3+3t^2-4t+5$	3 / ✘	$t^3-8t^2+16t-20$	1 / ✘
	10^n_{142}	$2t^3-3t^2+2t-1$	3 / ✘	$-13t^5+12t^4-13t^3+4t^2-17t+12$	3 / ✘
	10^n_{143}	t^3-3t^2+6t-7	3 / ✘	$t^5-4t^4+15t^3-28t^2+45t-48$	1 / ✘
	10^n_{144}	$-3t^2+10t-13$	2 / ✘	$10t^3-44t^2+80t-96$	2 / ✘
	10^n_{145}	t^2+t-3	2 / ✘	$2t^3+8t^2+6t-8$	2 / ✘
	10^n_{146}	$2t^2-8t+13$	2 / ✘	$t^3-8t^2+21t-28$	1 / ✘
	10^n_{147}	$-2t^2+7t-9$	2 / ✘	$-3t^3+12t^2-15t+12$	1 / ✘
	10^n_{148}	t^3-3t^2+7t-9	3 / ✘	$t^5-4t^4+18t^3-36t^2+62t-68$	2 / ✘
	10^n_{149}	$-t^3+5t^2-9t+11$	3 / ✘	$2t^5-18t^4+55t^3-104t^2+149t-164$	2 / ✘
	10^n_{150}	$-t^3+4t^2-6t+7$	3 / ✘	$-2t^5+12t^4-26t^3+38t^2-45t+44$	2 / ✘
	10^n_{151}	$t^3-4t^2+10t-13$	3 / ✘	$-t^5+6t^4-21t^3+42t^2-66t+72$	2 / ✘
	10^n_{152}	$t^4-t^3-t^2+4t-5$	4 / ✘	$4t^7-7t^5+18t^4-7t^3-12t^2+45t-52$	4 / ✘
	10^n_{153}	t^3-t^2-t+3	3 / ✔	$t^5-2t^4+t^3+2t^2-t$	2 / ✘
	10^n_{154}	t^3-4t+7	3 / ✘	$-3t^5-6t^4+13t^3-47t+68$	3 / ✘
	10^n_{155}	$-t^3+3t^2-5t+7$	3 / ✔	$-2t^3+12t^2-22t+28$	2 / ✘
	10^n_{156}	t^3-4t^2+8t-9	3 / ✘	$t^5-6t^4+19t^3-30t^2+33t-32$	1 / ✘
	10^n_{157}	$-t^3+6t^2-11t+13$	3 / ✘	$-2t^5+22t^4-78t^3+148t^2-218t+240$	2 / ✘
	10^n_{158}	$-t^3+4t^2-10t+15$	3 / ✘	$2t^2-7t+12$	2 / ✘
	10^n_{159}	$t^3-4t^2+9t-11$	3 / ✘	$t^5-6t^4+26t^3-60t^2+98t-112$	1 / ✘
	10^n_{160}	$-t^3+4t^2-4t+3$	3 / ✘	$-2t^5+12t^4-20t^3+14t^2-16t+12$	2 / ✘
	10^n_{161}	t^3-2t+3	3 / ✘	$3t^5+6t^4-3t^3+4t^2+14t-12$	3 / ✘
	10^n_{162}	$-3t^2+9t-11$	2 / ✘	$10t^3-38t^2+58t-68$	2 / ✘
	10^n_{163}	$t^3-5t^2+12t-15$	3 / ✘	$-t^5+8t^4-30t^3+62t^2-89t+96$	1, 2 / ✘
	10^n_{164}	$3t^2-11t+17$	2 / ✘	$t^3-10t^2+29t-40$	1 / ✘
	10^n_{165}	$-2t^2+10t-15$	2 / ✘	$-5t^3+50t^2-146t+196$	2 / ✘